Domino Snake Problems on Groups

Nicolás Bitar, Nathalie Aurbun

October 6, 2023

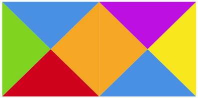
Séminaire des Mathématiques Discrètes - Liège

Wang Tiles

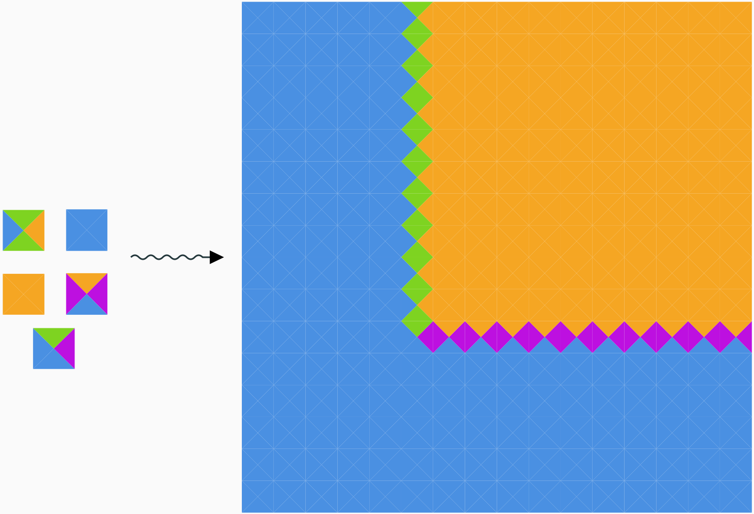
- Our story begins with Wang tiles:

First introduced by Hao Wang in 1961.

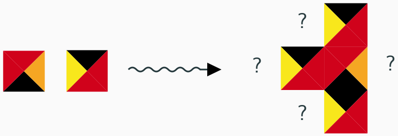
- They can be placed side by side if they share the same color along their common border.



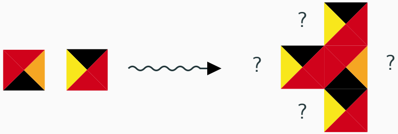
Tilings



Tilings



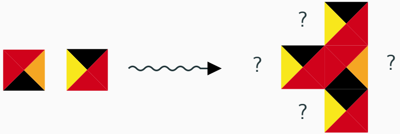
Tiling?



Domino Problem

Given a finite set τ of Wang tiles, does τ tile the plane?

Tiling?



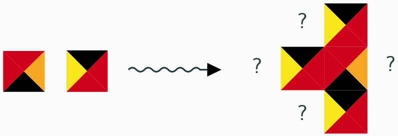
Domino Problem

Given a finite set τ of Wang tiles, does τ tile the plane?

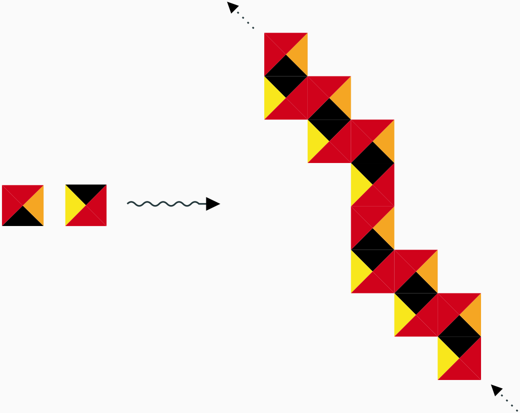
Theorem (Berger '64)

The Domino Problem is undecidable (Π_1^0-comp.).

Snakes?



Snakes!



New Problem

An *a priori* weaker version of the Domino Problem:

Infinite Snake

Given a finite Wang tileset τ, does there exist a snake tiled by τ?

New Problem

An *a priori* weaker version of the Domino Problem:

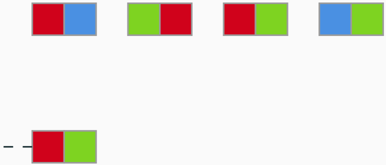
Infinite Snake

Given a finite Wang tileset τ, does there exist a snake tiled by τ?

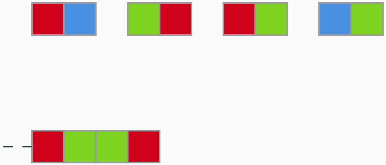
Theorem (Adleman, J. Kari, L. Kari, Reishus '02)

The infinite snake problem in $\mathbb{Z}^2$ is undecidable (Π_1^0-comp).

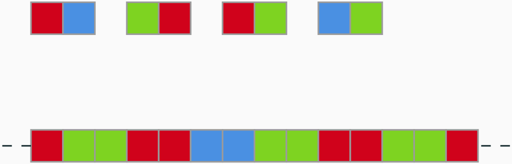
Lineland



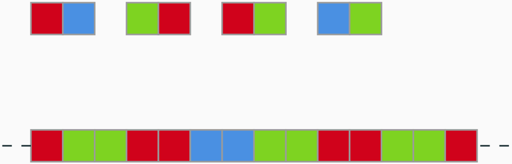
Lineland



Lineland



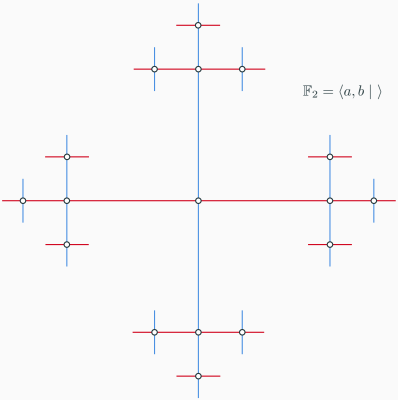
Lineland



In $\mathbb{Z}$ both the Domino Problem and Infinite Snake Problem are decidable. Why?

Where is the Undecidability?

As was done for the Domino Problem, we study our problem in a particular class of graphs: Cayley graphs of finitely generated infinite groups.

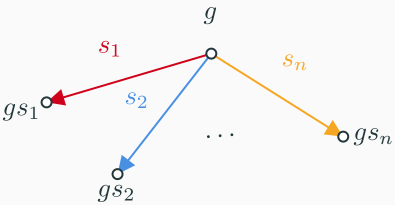


These graphs are infinite, locally finite, regular, transitive, edge labelled.

Groups as Graphs

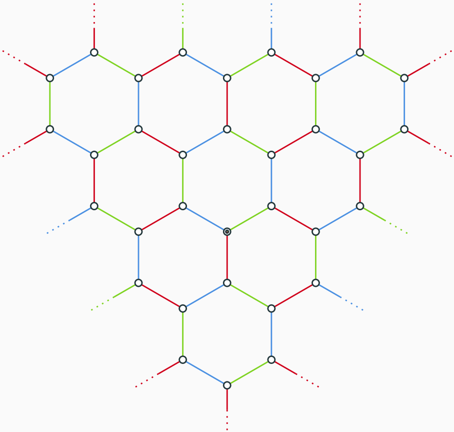
A Cayley graph is defined from a group G along with a finite generating set S:

- Vertices are elements of G,
- There is an edge from g to h if $h = gs^{\pm 1}$.

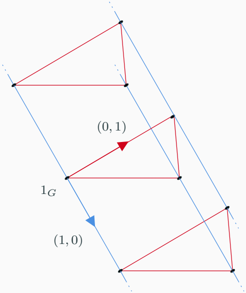


Examples

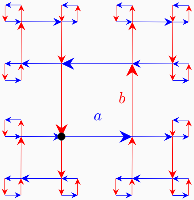
$$\langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^3, (ac)^3 \rangle$$



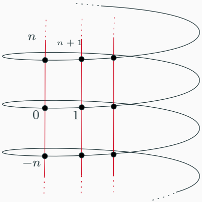
More Examples



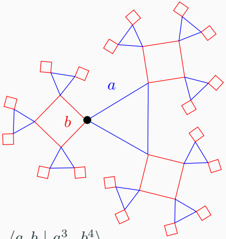
$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$



$\langle a, b \mid (ab)^2 \rangle$



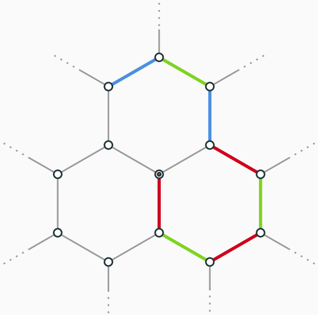
$(\mathbb{Z},\ \{1, n\})$



$\langle a, b \mid a^3,\ b^4 \rangle$

Given a Cayley graph for a group G with generating set S, there is a correspondence between paths and words in $(S \cup S^{-1})^*$.

$ababacbc \longrightarrow$



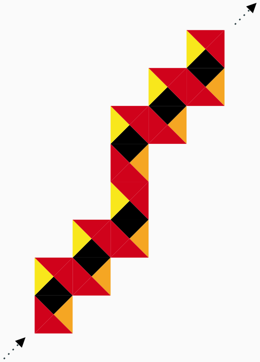
For instance, cycles are described by the set

$$\mathsf{WP}(G, S) = \{w \in (S \cup S^{-1})^* \colon \; w =_G \varepsilon\}.$$

Formal snakes

Let τ be a Wang tileset. A τ-snake is a pair of functions (ω, ζ):

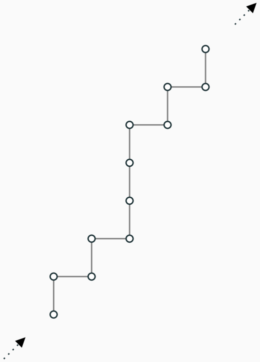
- (Skeleton) $\omega : \mathbb{Z} \to G$ injective s.t. $\omega(i+1)\omega(i)^{-1} \in S \cup S^{-1}$,
- (Scales) $\zeta : \mathbb{Z} \to \tau$ respecting local adjacency rules.



Formal snakes

Let τ be a Wang tileset. A τ-snake is a pair of functions (ω, ζ):

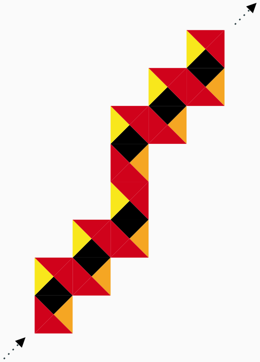
- (Skeleton) $\omega : \mathbb{Z} \to G$ injective s.t. $\omega(i+1)\omega(i)^{-1} \in S \cup S^{-1}$,
- (Scales) $\zeta : \mathbb{Z} \to \tau$ respecting local adjacency rules.



Formal snakes

Let τ be a Wang tileset. A τ-snake is a pair of functions (ω, ζ):

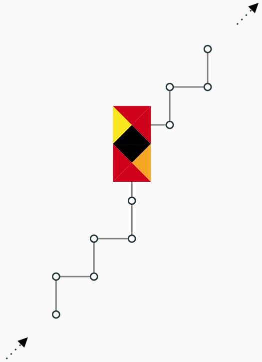
- (Skeleton) $\omega : \mathbb{Z} \to G$ injective s.t. $\omega(i+1)\omega(i)^{-1} \in S \cup S^{-1}$,
- (Scales) $\zeta : \mathbb{Z} \to \tau$ respecting local adjacency rules.



Formal snakes

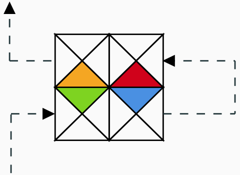
Let τ be a Wang tileset. A τ-snake is a pair of functions (ω, ζ):

- (Skeleton) $\omega : \mathbb{Z} \to G$ injective s.t. $\omega(i+1)\omega(i)^{-1} \in S \cup S^{-1}$,
- (Scales) $\zeta : \mathbb{Z} \to \tau$ respecting local adjacency rules.



Strong snakes

In particular, we do not care if adjacent, but not sequentially adjacent, tiles match.



Results

Our investigation went along the following lines:

1. **Reducing the problem from one group to another**:
   Virtually nilpotent groups that are not virtually $\mathbb{Z}$ admit a Cayley graph with undecidable infinite snake problem.

2. **Finding decidability by restricting possible skeletons**:
   The infinite snake problem on $\mathbb{Z}^2$ becomes decidable when considering geodesic skeletons.

3. **Expressing the problem in MSO logic**:
   Virtually free groups have decidable infinite snake problem on all Cayley graphs.

Results

Our investigation went along the following lines:

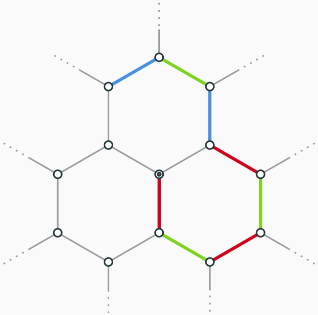
1. **Reducing the problem from one group to another**:
   Virtually nilpotent groups that are not virtually $\mathbb{Z}$ admit a Cayley graph with undecidable infinite snake problem.

2. **Finding decidability by restricting possible skeletons**:
   The infinite snake problem on $\mathbb{Z}^2$ becomes decidable when considering geodesic skeletons.

3. **Expressing the problem in MSO logic**:
   Virtually free groups have decidable infinite snake problem on all Cayley graphs.

Skeletons

Given a Cayley graph for a group G with generating set S, there is a correspondence between paths and words in $(S \cup S^{-1})^*$.

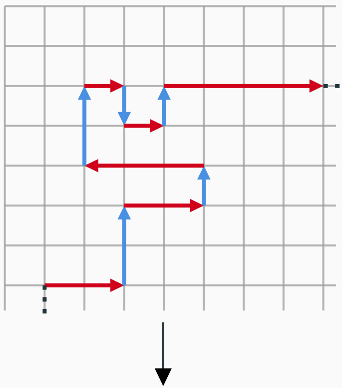
$ababacbc$ ⟶ 

For instance, cycles are described by the set

$$\mathsf{WP}(G, S) = \{w \in (S \cup S^{-1})^* \colon\ w =_G \varepsilon\}.$$

Voyage to Lineland

This allows us to understand skeletons as bi-infinite words without loops.



$\ldots a\ a\ b\ b\ a\ a\ b\ a^{-1}\ a^{-1}\ a^{-1}\ b\ b\ a\ b^{-1}\ a\ b\ a\ a\ a\ a \ldots$

Skeletons

Let G be a f.g. group with S a set of generators.

The skeleton set of the pair (G, S) is

$$X_{G,S} = \{x \in (S \cup S^{-1})^{\mathbb{Z}} \mid \forall w \sqsubseteq x,\ w \notin \mathsf{WP}(G,S)\},$$

For instance:

$$X_{\mathbb{Z}^2,\{a,b\}} = \left\{x \in \{a^{\pm 1}, b^{\pm 1}\}^{\mathbb{Z}} \colon\ \forall w \sqsubseteq x, |w|_a \neq |w|_{a^{-1}} \vee |w|_b \neq |w|_{b^{-1}} \right\}.$$

Y Skeletons?

Let $Y \subseteq X_{G,S}$ be a subset of skeletons.

Y-Snake Problem

Given a Wang tileset τ, does there exist a snake tiled by τ whose skeleton is contained in Y?

Lemma

If Y is a sofic shift, then the Y-snake problem is decidable.

Snakes Fill the Emptiness

Lemma

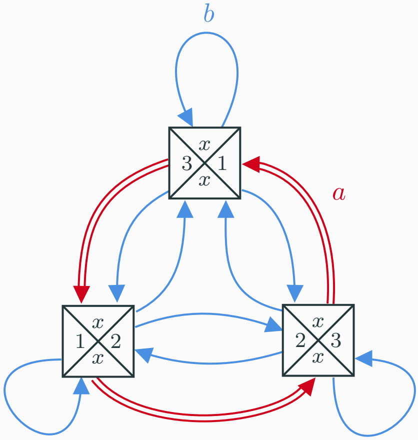
If Y is sofic, then the Y-snake problem is decidable.

Proof sketch:

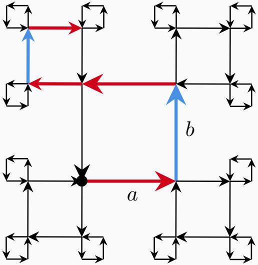
- Create a $(S \cup S^{-1})$-edge labeled graph from the Wang tileset τ, denoted Γ.
- This defines a sofic set Y_Γ of labels of bi-infinite paths on Γ.
- There is a Y-snake iff $Y \cap Y_\Gamma \neq \varnothing$.

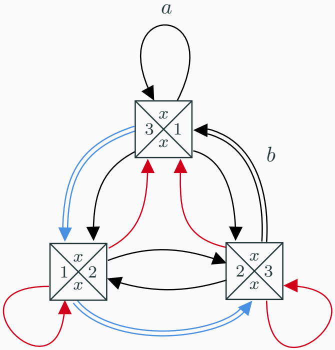
Snakes Fill the Emptiness

Example of Γ:



Example: $\langle a, b \mid (ab)^2 \rangle$





Example: the Free Group

Proposition

The infinite snake problem is decidable in $\mathbb{F}_n$, for the canonical generating set.

Example: the Free Group

Proposition

The infinite snake problem is decidable in $\mathbb{F}_n$, for the canonical generating set.

Proof:

- Any word from $\mathrm{WP}(\mathbb{F}_n, S)$ must contain a factor of the form ww^{-1} with $w \in (S \cup S^{-1})^*\}$,

Example: the Free Group

Proposition

The infinite snake problem is decidable in $\mathbb{F}_n$, for the canonical generating set.

Proof:

- Any word from $\text{WP}(\mathbb{F}_n, S)$ must contain a factor of the form ww^{-1} with $w \in (S \cup S^{-1})^*\}$,
- Then, $\{ss^{-1} \mid s \in S \cup S^{-1}\}$ is a set of forbidden patterns for $X_{\mathbb{F}_n,S}$,

Example: the Free Group

Proposition

The infinite snake problem is decidable in $\mathbb{F}_n$, for the canonical generating set.

Proof:

- Any word from WP$(\mathbb{F}_n, S)$ must contain a factor of the form ww^{-1} with $w \in (S \cup S^{-1})^*\}$,
- Then, $\{ss^{-1} \mid s \in S \cup S^{-1}\}$ is a set of forbidden patterns for $X_{\mathbb{F}_n,S}$,
- This means $X_{\mathbb{F}_n,S}$ is an SFT.

Example: the Free Group

Proposition

The infinite snake problem is decidable in $\mathbb{F}_n$, for the canonical generating set.

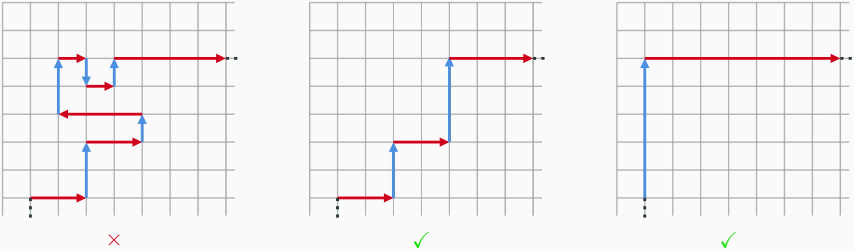
Proof:

- Any word from $\mathrm{WP}(\mathbb{F}_n, S)$ must contain a factor of the form ww^{-1} with $w \in (S \cup S^{-1})^*\}$,
- Then, $\{ss^{-1} \mid s \in S \cup S^{-1}\}$ is a set of forbidden patterns for $X_{\mathbb{F}_n,S}$,
- This means $X_{\mathbb{F}_n,S}$ is an SFT.
- By the previous Lemma, the problem is decidable.

Geodesic Skeletons

A skeleton of particular interest is the set of bi-infinite geodesics:

$$X^g_{G,S} = \{x \in X_{G,S} \colon \ \forall w \sqsubseteq x, w' =_G w \colon \ |w| \leq |w'|\} \subseteq X_{G,S}$$



$X^g_{\mathbb{Z}^2,\{a,b\}}$ is sofic!

No Curves

Theorem

The Y-snake problem on $\mathbb{Z}^2$ is decidable for:

- Geodesic skeletons,
- 3 or less canonical directions,

or in general:

Theorem

If a group G with generating set S has a set of geodesics that is regular, then its geodesic snake problem is decidable.

This happens with many classes, such as abelian groups and hyperbolic groups, for all generating sets.

Results

Our investigation went along the following lines:

1. **Reducing the problem from one group to another**:
   Virtually nilpotent groups that are not virtually $\mathbb{Z}$ admit a Cayley graph with undecidable infinite snake problem.
2. **Finding decidability by restricting possible skeletons**:
   The infinite snake problem on $\mathbb{Z}^2$ becomes decidable when considering geodesic skeletons.
3. **Expressing the problem in MSO logic**:
   Virtually free groups have decidable infinite snake problem on all Cayley graphs.

MSO logic and Trees

$$\text{Virtually free} \iff \text{Finite tree-width} \iff \text{Decidable MSO logic}$$

[Muller, Schupp '85] [Kuskey, Lohrey '05]

Therefore, if the infinite snake problem can be expressed in MSO logic, it is decidable for this class of groups.

Theorem

The infinite snake problem is decidable on virtually free groups, for every generating set.

MSO logic and Trees

MSO logic of an S labelled graph Γ consists in:

- Variables are subsets of vertices, along with the constant set $\{v_0\}$,
- an operation $P \cdot s$, representing all the vertices reached from P when reading s,
- Boolean operations $\vee, \wedge, \subseteq, \neg, \ldots$ and quantifiers $\forall, \exists$

Theorem

The infinite snake problem is decidable on virtually free groups, for every generating set.

Results

Our investigation went along the following lines:

1. **Reducing the problem from one group to another**:
   Virtually nilpotent groups that are not virtually $\mathbb{Z}$ admit a Cayley graph with undecidable infinite snake problem.

2. **Expressing the problem in MSO logic**:
   Virtually free groups have decidable infinite snake problem on all Cayley graphs.

3. **Finding decidability by restricting possible skeletons**:
   The infinite snake problem on $\mathbb{Z}^2$ becomes decidable when considering geodesic skeletons.

Thank you for listening!

Spooky appendix

Let $\mathcal{P}$ be a group property (abelian, nilpotent, free, etc). We say a group is virtually $\mathcal{P}$ if it contains a finite index subgroup satisfying $\mathcal{P}$.

For instance, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is virtually $\mathbb{Z}$, the Honeycomb group ($\mathbb{Z}^2 \rtimes S_3$) is virtually nilpotent, $SL(2, \mathbb{Z})$ is virtually free.

Virtually nilpotent $\iff$ Polynomial growth rate,

[Gromov '81]

Virtually free $\iff$ Finite tree-width.

[Muller, Schupp '85]

Very Undecidable Skeletons

Corollary

If Y is an effective $\mathbb{Z}$-subshift, then the Y-skeleton snake problem is Π^0_1.

Very Undecidable Skeletons

Corollary

If Y is an effective $\mathbb{Z}$-subshift, then the Y-skeleton snake problem is Π_1^0.

What happens if Y is not closed?

Very Undecidable Skeletons

Corollary

If Y is an effective $\mathbb{Z}$-subshift, then the Y-skeleton snake problem is Π_1^0.

What happens if Y is not closed?

Theorem (Ebbinghaus '86)

Let us have the $\mathbb{Z}^2$-skeletal subset

$$Y = \{x \in \{a, b, a^{-1}, b^{-1}\}^{\mathbb{Z}} \mid x \text{ is not eventually a line.}\}$$

Then the Y-skeleton snake problem is Σ_1^1-complete.

Snakes in Space!

What happens in higher dimensions?

Proposition

The infinite snake problem in $\mathbb{Z}^d$, with $d \geq 2$ is undecidable for all generating sets.