Snakes, SAWs and Symbolic Dynamics

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joint work with Nathalie Aubrun

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LABORATOIRE INTERDISCIPLINAIRE DES SCIENCES DU NUMÉRIQUE

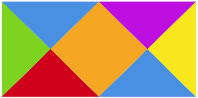
Complexity of Simple Dynamical Systems in honor of Jarkko Kari's 60th birthday

Wang Tiles

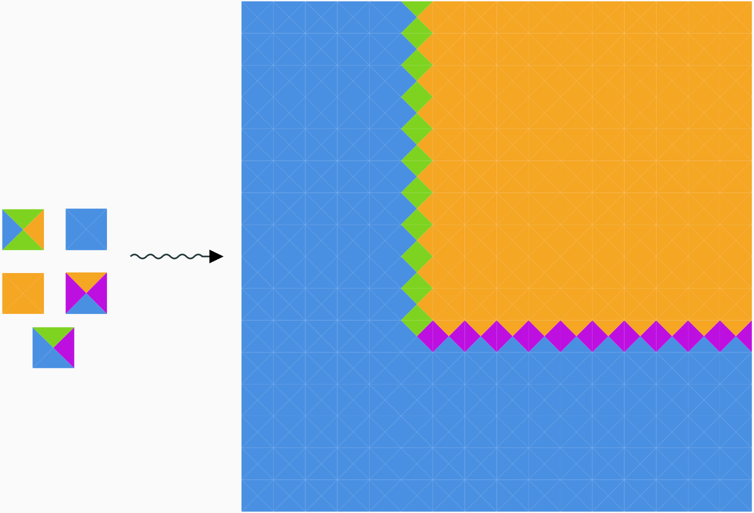
- Our story begins with Wang tiles:

First introduced by Hao Wang in 1961.

- They can be placed side by side if they share the same color along their common border.



Tilings



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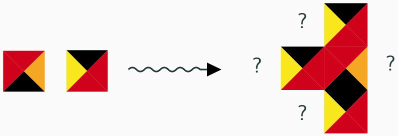
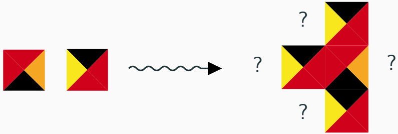


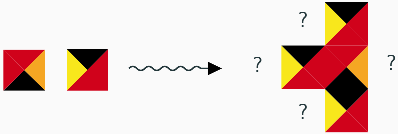
Tiling?



Domino Problem

Given a finite set τ of Wang tiles, does τ tile the plane?

Tiling?



Domino Problem

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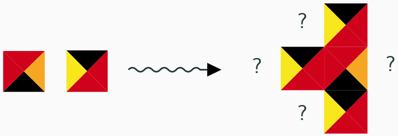
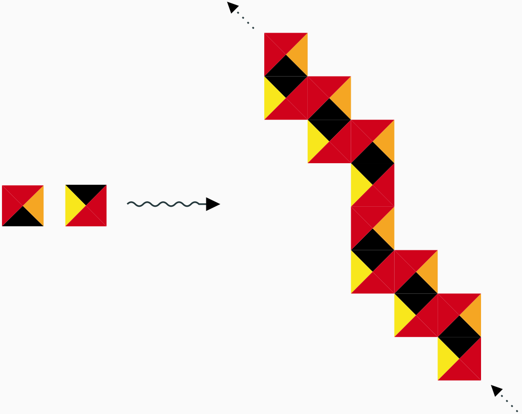
Theorem (Berger '64)

The Domino Problem is undecidable (Π^0_1-comp.).

Snakes?



Snakes!



New Problem: Snakes on a Plane

An *a priori* weaker version of the Domino Problem:

Infinite Snake

Given a finite Wang tileset τ, does there exist a snake tiled by τ?

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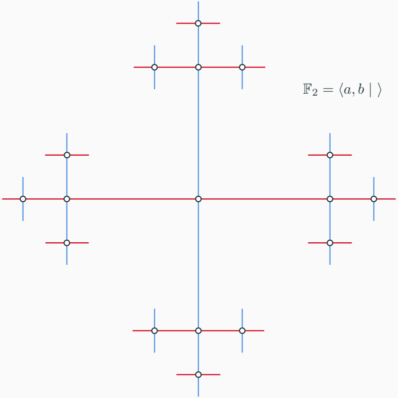
Theorem (Adleman, J. Kari, L. Kari, Reishus '02)

The infinite snake problem in $\mathbb{Z}^2$ is undecidable (Π_1^0-comp).

Nevertheless, in $\mathbb{Z}$ both the Domino Problem and Infinite Snake Problem are decidable. Why?

Where is the Undecidability?

As was done for the Domino Problem, we study our problem in a particular class of graphs: Cayley graphs of finitely generated infinite groups.

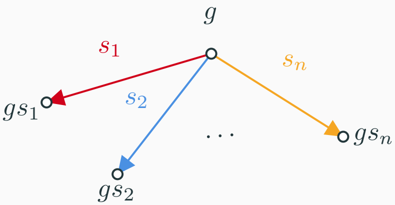


These graphs are infinite, locally finite, regular, transitive, edge labelled.

Groups as Graphs

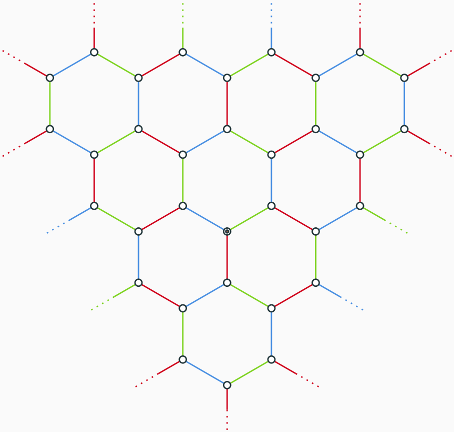
A Cayley graph is defined from a group G along with a finite symmetric generating set S:

- Vertices are elements of G,
- There is an edge from g to h if $h = gs^{\pm 1}$.

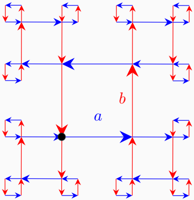


Examples

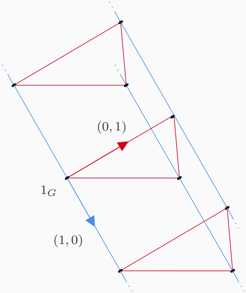
$$\mathbb{H} = \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^3, (ac)^3 \rangle$$



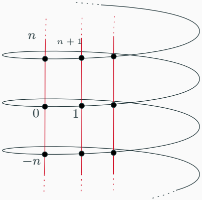
More Examples



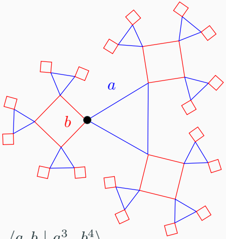
$\langle a, b \mid (ab)^2 \rangle$



$\mathbb{Z} \times C_3$



$(\mathbb{Z},\ \{1, n\})$

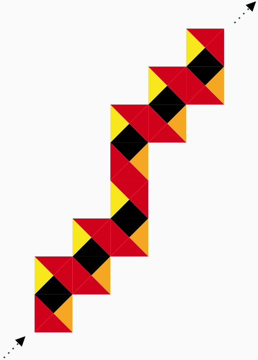


$\langle a, b \mid a^3,\ b^4 \rangle$

Formal Snakes

Let τ be a tileset. A τ-snake is a pair of functions (ω, ζ):

- (Skeleton) $\omega : \mathbb{Z} \to G$ injective s.t. $\omega(i)^{-1}\omega(i+1) \in S$,
- (Scales) $\zeta : \mathbb{Z} \to \tau$ respecting local adjacency rules.



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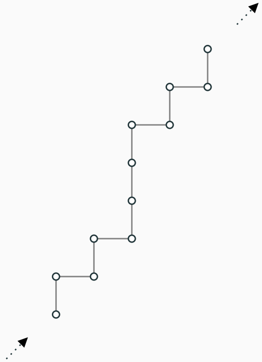

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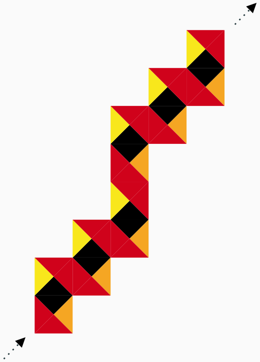

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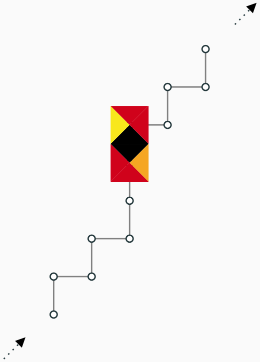
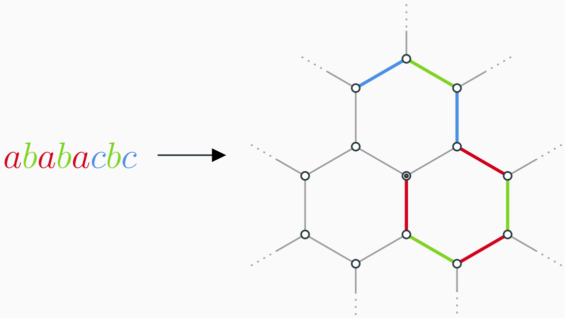


Words and Paths

Given a Cayley graph for a group G with generating set S, there is a correspondence between paths and words in S^*.

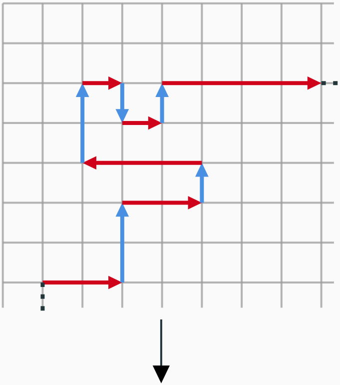


For instance, cycles are described by the set

$$\mathsf{WP}(G, S) = \{w \in S^* \mid w =_G \varepsilon\}.$$

Skeletons

This allows us to understand skeletons as bi-infinite words without loops.



$\ldots a\ a\ b\ b\ a\ a\ b\ a^{-1}\ a^{-1}\ a^{-1}\ b\ b\ a\ b^{-1}\ a\ b\ a\ a\ a\ a \ldots$

Skeletons

Let G be a finitely generated group with S a set of generators.

The skeleton of G with respect to S is

$$\mathbb{X}_{G,S} = \{x \in S^{\mathbb{Z}} \mid \forall w \sqsubseteq x,\ w \notin \mathsf{WP}(G,S)\},$$

Examples:

$$\mathbb{X}_{\mathbb{Z}^2,\{a^{\pm1},b^{\pm1}\}} = \left\{x \in \{a^{\pm1},b^{\pm1}\}^{\mathbb{Z}}\colon\ \forall w \sqsubseteq x, |w|_a \neq |w|_{a^{-1}} \vee |w|_b \neq |w|_{b^{-1}}\right\}.$$

For the infinite dihedral group $\mathcal{D}_\infty = \langle a, b \mid a^2, b^2\rangle$,

$$\mathbb{X}_{\mathcal{D}_\infty,\{a,b\}} = \{(ab)^\infty, (ba)^\infty\}.$$

Decidability

Theorem (Aubrun, B. '23)

If $\mathbb{X}_{G,S}$ is sofic, then the infinite snake problem for (G, S) is decidable.

Questions

When is $\mathbb{X}_{G,S}$ sofic? SFT? Effective?

What are its periodic points?

What is its entropy?

Self-Avoiding Walks

- A self-avoiding walk (SAW) is a path on a graph that visits each vertex at most once.
- For $c(n)$ = number of SAWs of length n,

$$\mu(G, S) = \lim_{n\to\infty} \sqrt[n]{c(n)}$$

is know as the connective constant of the Cayley graph.

Examples:

- (Duminil-Copin, Smirnov '12) $\mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}$,
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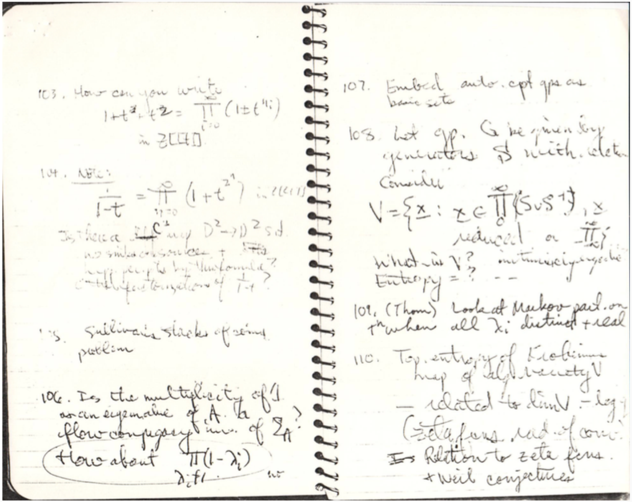
Best approx so far (Jacobsen, Scullard, Guttman '16):

$$\mu(\mathbb{Z}^2) \approx 2.63815853032790(3)$$

More Motivations

In fact, $\mathbb{X}_{G,S}$ is the set of labels of bi-infinite SAWs!

It also appears as Problem 108 in Rufus Bowen's notebook of problems:



Sofic Snakes and SAWs

Theorem (Aubrun, B. ’24)

G admits S such that $\mathbb{X}_{G,S}$ is sofic iff G is a plain group, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}$.

Proposition (Aubrun, B. ’24)

For every group G, there exists S such that $\mathbb{X}_{G,S}$ is not sofic.

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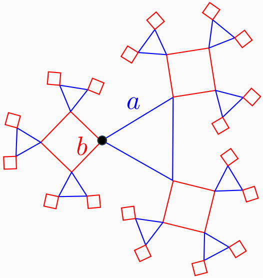
A group G is said to be plain if there are finite groups $\{G_i\}_{i=1}^m$ and $n \in \mathbb{N}$ such that

$$G \simeq \left(\mathop{\ast}_{i=1}^{m} G_i \right) \ast \mathbb{F}_n.$$



$$G = \mathbb{Z}/3\mathbb{Z} \ast \mathbb{Z}/4\mathbb{Z},$$
$$= \langle a, b \mid a^3, b^4 \rangle.$$

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Theorem (Aubrun, B. '24)

G admits S s.t. $\mathbb{X}_{G,S}$ is an SFT iff G is a plain group.

Step 1: Torsion

First off, by Kőnig's lemma,

$$\mathbb{X}_{G,S} = \varnothing \iff G \text{ is finite.}$$

Then,

Theorem (Aubrun, B. '24)

G is a torsion group iff $\mathbb{X}_{G,S}$ is aperiodic for (any) all generating sets S.

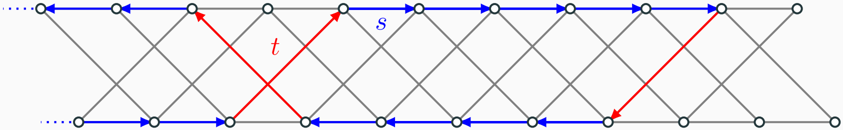
Torsion groups can never have sofic skeletons!

Oh No

Proposition (Aubrun, B. '24)

For every group G, there exists S such that $\mathbb{X}_{G,S}$ is not sofic.

We can use the Pumping Lemma to show being sofic depends on the generating set. For a torsion-free element $g \in G$ we add $s = g^2$ and $t = g^3$ to S so we can find a copy of Cay$(\mathbb{Z}, \{\pm 2, \pm 3\})$.



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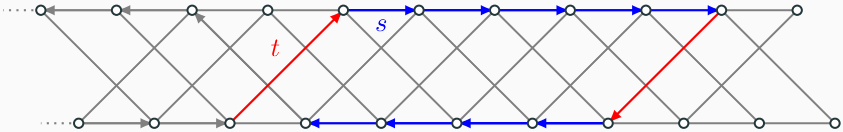
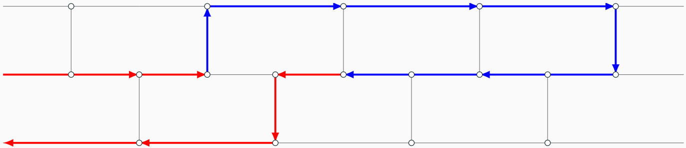


$\forall n \in \mathbb{N}$, $\mathcal{L}(\mathbb{X}_{G,S})$ contains the configuration $ts^{n+1}t^{-1}s^{-n}$ on which we use the Pumping Lemma.

Step 2: Ends

- First we show that all ends are thin and of size at most 2.



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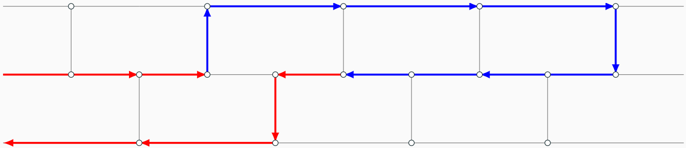
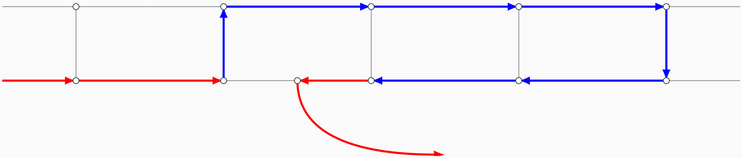


- If there is an end of size 2, then G is virtually $\mathbb{Z}$.



Step 3: Plain and Virtually $\mathbb{Z}$

Lemma (Aubrun, B. '24)

If $G \notin \{\mathbb{Z},\ \mathcal{D}_\infty,\ \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ \mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}\}$ and virtually $\mathbb{Z}$, then every Cayley graph of G has ends of size ≥ 3.

Theorem (Haring-Smith '83 + Lindorfer, Woess '20)

G is plain iff it admits a Cayley graph with ends of size 1.

Rigolo Properties

- $h_{top}(\mathbb{X}_{G,S}) = \log(\mu(G,S))$,
- G recursively presented $\implies \mathbb{X}_{G,S}$ is effective $\forall S$,
- $\mathbb{X}_{G,S}$ minimal $\implies$ all proper quotients of G are finite,
- There exists G s.t. for every S, $\mathbb{X}_{G,S}$ is effective and has no computable points,
- For some groups and generators (including $\mathbb{Z}^d$, $d \geq 2$),

$$h_{top}(\mathbb{X}_{G,S}) = \lim_{n\to\infty} \frac{\log(q(n))}{n},$$

for $q(n) =$ number of periodic points of period n.

Thank you for listening!