The k-SAT Problem on Groups

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Generalizing SAT

Let G be a f.g. group and a symmetric generating set S. Take words $\{w_{ij}\} \subseteq S^*$ and define the formula

$$\phi = \bigwedge_{i=1}^{m} ((w_{i1})' \vee ... \vee (w_{ik})'),$$

where w' represents w or the negation $\neg w$.

The **k-SAT problem for G** asks, given a formula with k literals ϕ and $\{u_i\}_{i=1}^{n} \subseteq S^*$, if there is an assignation of truth values $\alpha : G \to \{0,1\}$ such that

$$\bigwedge_{h\in H} \bigwedge_{i=1}^{m} (\alpha(hg_{i1})' \vee ... \vee \alpha(hg_{ik})') = 1,$$

where $H = \langle \overline{u}_1, ..., \overline{u}_n \rangle$, and $g_{ij} = \overline{w}_{ij}$.

Example on $\mathbb{Z}^2$

For $\mathbb{Z}^2 = \langle \mathtt{a}, \mathtt{b} \mid [\mathtt{a}, \mathtt{b}] \rangle$, take the formula:

$$\phi = (\neg 1_{\mathbb{Z}^2} \vee \mathtt{a} \vee \mathtt{b}),$$

and $H = (2\mathbb{Z})^2$:

1	0	1	1	1	1	0	0	0
0	1	1	0	0	1	1	1	0
0	0	0	1	1	0	1	0	0
0	0	0	0	0	1	0	1	1
0	1	0	0	0	0	0	0	1
1	1	0	1	0	0	0	0	0
1	1	1	1	1	1	1	1	1
0	0	1	1	1	0	0	0	0
1	0	1	1	1	1	0	0	0

Reductions

The **subgroup membership problem** of G asks, given words $w, \{u_i\}_{i=1}^{n}$ in S^*, if $\overline{w} \in \langle \overline{u}_1, ..., \overline{u}_n \rangle$

Proposition

The subgroup membership problem of G reduces to 2-SAT(G).

The **Domino Problem** on G asks, given an alphabet A and a set of forbidden patterns $\mathcal{F} \subseteq A^2 \times S$, if there exists a map $x : G \to A$ such that $(x(g), x(gs), s) \notin \mathcal{F}$.

Proposition

If G has decidable subgroup membership problem, then k-SAT(G) reduces to DP(G) for $k \geq 2$.

This implies **virtually free** groups have decidable k-SAT.

Main Result

Theorem

If G admits a strict finite index subgroup $H \lneq G$ such that $H \simeq G$, then DP(G) reduces to 3-SAT(G).

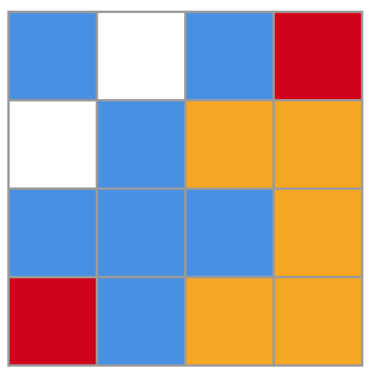
For an alphabet of size n, take a subgroup of index $\geq \lceil \log_2(n) \rceil$ and code each letter. If $G = \mathbb{Z}^2$ and $A = \{\ ,\ ,\ ,\ \}$, take $H = \mathbb{Z} \times 2\mathbb{Z}$:

$$\mapsto \begin{pmatrix}0\\0\end{pmatrix} \longleftrightarrow \phi_1 = \neg 1_{\mathbb{Z}^2} \wedge \neg \mathtt{b},$$

$$\mapsto \begin{pmatrix}0\\1\end{pmatrix} \longleftrightarrow \phi_2 = 1_{\mathbb{Z}^2} \wedge \neg \mathtt{b},$$

$$\mapsto \begin{pmatrix}1\\0\end{pmatrix} \longleftrightarrow \phi_3 = \neg 1_{\mathbb{Z}^2} \wedge \mathtt{b},$$

$$\mapsto \begin{pmatrix}1\\1\end{pmatrix} \longleftrightarrow \phi_4 = 1_{\mathbb{Z}^2} \wedge \mathtt{b}.$$



$\longmapsto$

0	0	0	1
1	0	1	0
0	0	1	1
0	1	1	1
0	0	0	1
1	1	1	1
1	0	1	1
0	1	1	1

If $\mathcal{F} = \{(\ ,\ , \mathtt{b})\}$, the formula coding the problem is given by,

$$\left(\bigvee_{i=1}^{4} \phi_i(1_{\mathbb{Z}^2})\right) \wedge \left(\neg\phi_3(1_G) \vee \neg\phi_3(\mathtt{b}^2)\right).$$

which is then transformed into 3-CNF form.

Corollary

3-SAT(G) is undecidable for virtually $\mathbb{Z}^d$ groups for $d \geq 2$, the Heisenberg group, $BS(1,n)$, torus knot groups, free-by-cyclic groups $\mathbb{F}_n \rtimes_\theta \mathbb{Z}$, where θ has finite order in Out($\mathbb{F}_n$), lamplighter groups, and $\mathbb{Z}^d \rtimes GL(d, \mathbb{Z})$.

References

[1] Bitar, N., *Contributions to the Domino Problem: Seeding, Recurrence and Satisfiability*, 41st International Symposium on Theoretical Aspects of Computer Science (STACS 2024). Vol. 289. LIPIcs., pg.46-59, 2024.