Sofic Skeletons

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Skeleton Subshift

Given a f.g. group G and a symmetric generating set S we define the **skeleton subshift** as

$$\mathbb{X}_{G,S} = \{x \in S^{\mathbb{Z}} \mid \forall w \sqsubseteq x, w \notin \mathrm{WP}(G,S)\}.$$

We define its **language** as

$$\mathcal{L}(\mathbb{X}_{G,S}) = \{w \in S^* \mid \exists x \in \mathbb{X}_{G,S}, w \sqsubseteq x\}.$$

We say the skeleton $\mathbb{X}_{G,S}$ is **sofic** if $\mathcal{L}(\mathbb{X}_{G,S})$ is a regular language.

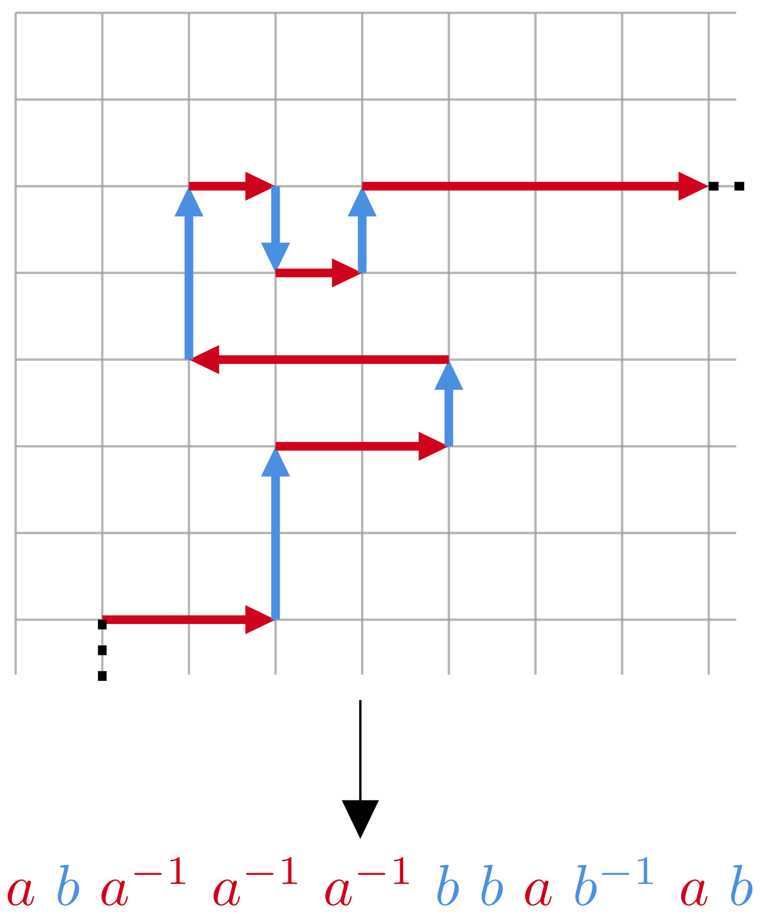
Pompes Funèbres Lemma

Our main tool is the **Pumping Lemma**:

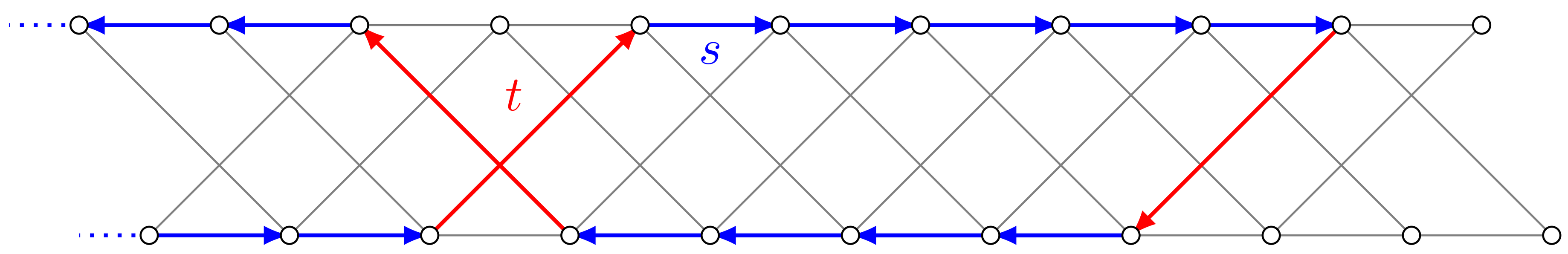
For L, a regular language, there exists $p > 0$ such that every word $w \in L$ with $|w| \geq p$ can be decomposed as $w = w'uv$ with $|u| > 1$ and $|uv| \leq p$ such that for all $n \in \mathbb{N}$, $w'u^n v \in L$.

Self-Avoiding Walks

$\mathbb{X}_{G,S}$ can be equivalently seen as the set of labels of bi-infinite **self-avoiding walks** on the Cayley graph $\mathrm{Cay}(G,S)$.



We can use the Pumping Lemma to show being sofic depends on the generating set. For a torsion-free element $g \in G$ we add $s = g^2$ and $t = g^3$ to S so we can find a copy of $\mathrm{Cay}(\mathbb{Z}, \{\pm 2, \pm 3\})$.



Then, for all $n \in \mathbb{N}$, $\mathcal{L}(\mathbb{X}_{G,S})$ contains the configuration $ts^{n+1}t^{-1}s^{-n}$ on which we use the Pumping Lemma.

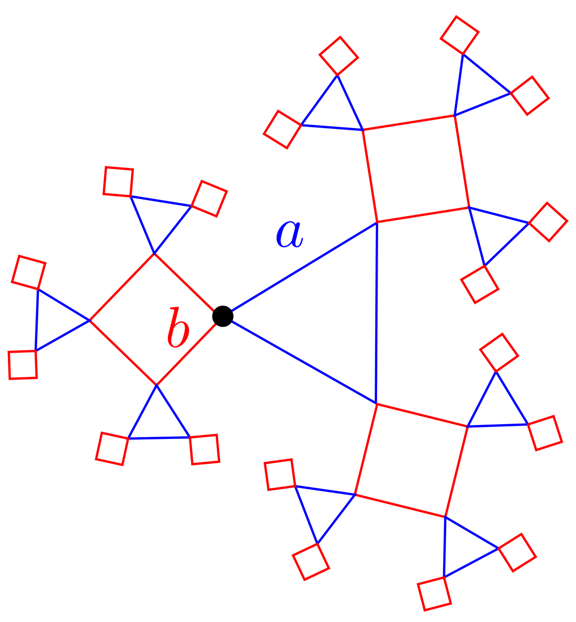
Proposition

For every group G there exists a generating set S such that $\mathbb{X}_{G,S}$ is not sofic.

Plain Groups

A group G is said to be **plain** if there are finite groups G_i such that

$$G \simeq \left(\mathop{\ast}_{i=1}^{m} G_i \right) * \mathbb{F}_n.$$



$\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/4\mathbb{Z} = \langle a, b \mid a^3, b^4 \rangle$

Theorem

Let G be a f.g. group. Then, $\exists S$ such that $\mathbb{X}_{G,S}$ is a **SFT** iff G is a plain group.

Characterization

We study thin and thick ends of $\mathrm{Cay}(G,S)$ to use the Pumping Lemma to obtain a characterization.

Theorem

Let G be a f.g. group. Then,

$$\exists S \text{ such that } \mathbb{X}_{G,S} \text{ is sofic}$$
$$\Updownarrow$$
$$G \text{ is a plain group, } \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \text{ or } \mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}.$$

To go further...

- The **Infinite Snake Problem** is decidable for (G,S) when $\mathbb{X}_{G,S}$ is sofic.
- The **entropy** of $\mathbb{X}_{G,S}$ is $\log(\mu(G,S))$, where $\mu(G,S)$ is the **connective constant** of $\mathrm{Cay}(G,S)$.

References

[1] Aubrun, N., Bitar, N., *Domino Snake Problems on Groups*, Proceedings of Fundamentals of Computation Theory (FCT 2023), pg.46-59, 2023.

[2] Lindorfer, C., Woess, W., *The Language of Self-Avoiding Walks*, Combinatorica 40.5, pp. 691–720, 2020.

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