Strongly Aperiodic SFTs on Generalized Baumslag-Solitar groups

Séminaire Dynamique et Probabilités - LAMFA

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LISN - Université Paris-Saclay

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Full-shift

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There is a natural action $G \curvearrowright A^G$ called the shift:

$$\sigma^g(x)_h = x_{g^{-1}h}.$$

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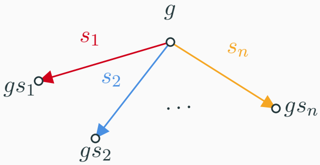
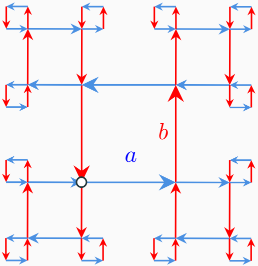
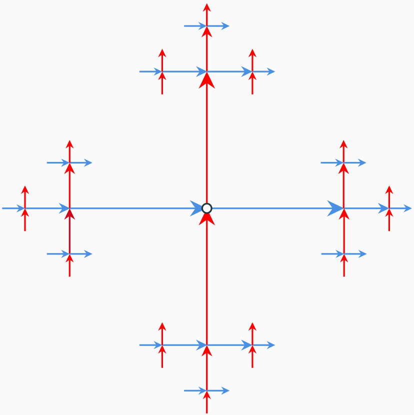


Cayley graph

$\langle a, b \mid (ab)^2 \rangle$



$\mathbb{F}_2 = \langle a, b \mid \rangle$



Subshifts

Definition

Let F be a set of patterns. We define a subshift as

$$X_F := \{x \in A^G \mid \text{no pattern in } F \text{ appears in } x\}$$

If $|F| < +\infty$, we say X_F is a subshift of finite type (SFT).

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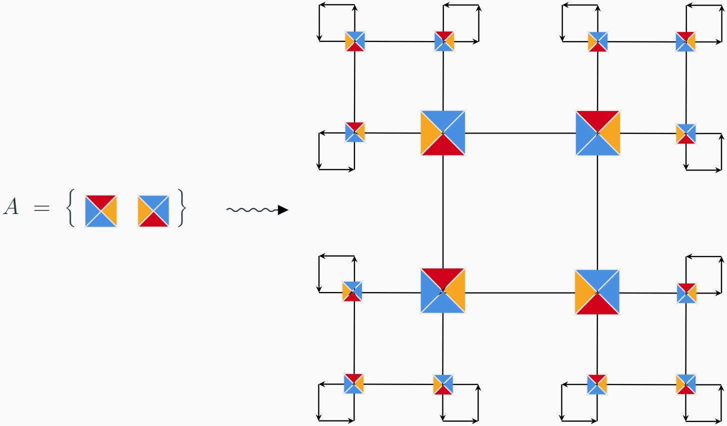
There is an alternative topological definition:

Proposition

X is a subshift iff it is a closed G-invariant subset of A^G.

Subshifts

Example of a configuration on $\langle a, b \mid (ab)^2 \rangle$:



Aperiodicity & Minimality

Definition

A *subshift* $X \neq \varnothing$ is strongly aperiodic if the group action is free.

$$\forall x \in X,\ \sigma^g(x) = x \implies g = 1_G.$$

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Definition

Let X be a subshift. We say X is minimal if the orbit of every configuration is dense.

Aperiodicity

Which groups admit strongly aperiodic subshifts?

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Every countable group G has a non-empty, strongly aperiodic subshift over the alphabet $\{0, 1\}$.

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Theorem (Bernhsteyn '19)

Every countable group G has a non-empty, strongly aperiodic minimal subshift.

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Theorem (Berger '66)

$\mathbb{Z}^2$ admits a strongly aperiodic SFT.

Orbit Coding

Proof of the Theorem (Kari '96):

- We code orbits of a simple dynamical system $\left([\frac{1}{10}, \frac{2}{5}]/_{\frac{1}{10}\sim\frac{2}{5}}, T\right)$.

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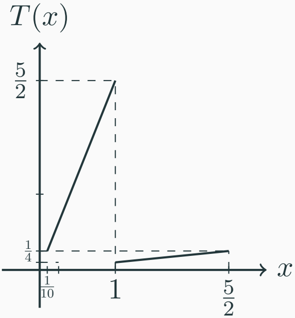

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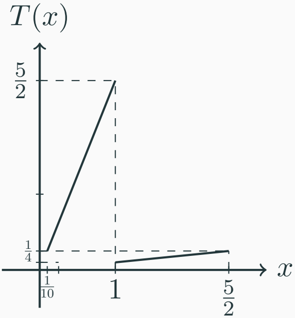

- T admits immortal points and is aperiodic.

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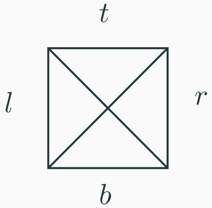
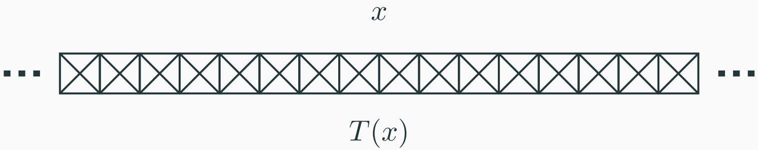


$$T(t) + r = b + l$$

Orbit coding



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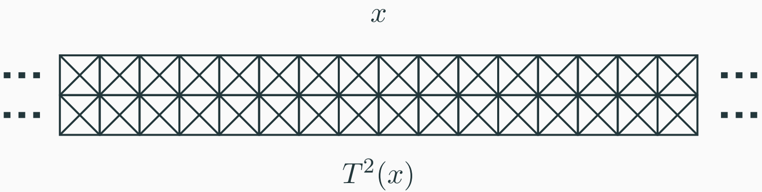

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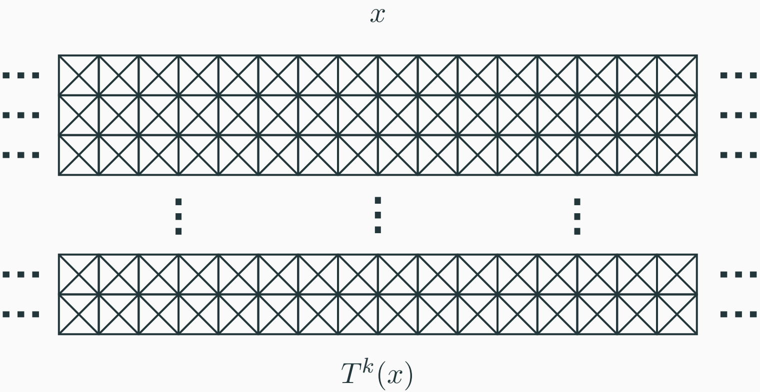

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Theorem (Jeandel '15)

If G has a strongly aperiodic SFT, then it has decidable word problem.

Conjecture

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G admits a strongly aperiodic SFT iff it is one-ended and has decidable word problem.

Classes of groups

It has been solved for some classes of groups!

- $\mathbb{Z}^d$ for $d \geq 2$ (Culik, Kari '96),
- Hyperbolic groups (Cohen, Goodman-Strauss, Rieck '17),
- Monster groups, $G \times \mathbb{Z}$ for G with property PA (Jeandel '15),
- $\mathbb{Z}^2 \rtimes H$ for H f.g. with decidable WP (Barbieri, Sablik '18),
- Groups with self-simulable 0-dim dynamics (Barbieri, Sablik, Salo '21),
- Residually finite BS groups (Esnay, Moutot '21)

Invariance properties

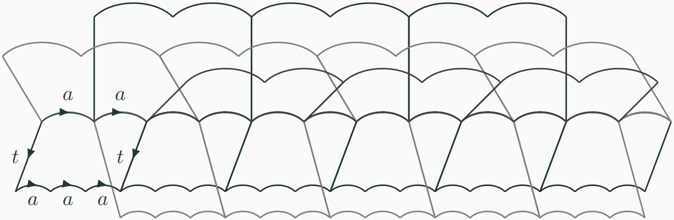
Theorem (Cohen '17)

Admitting strongly aperiodic SFTs is a quasi-isometry invariant for finitely presented groups.

Our contribution

Theorem (Aubrun, B., Huriot-Tattegrain '22)

All non-$\mathbb{Z}$ Generalized Baumslag-Solitar groups have undecidable domino problem and admit strongly aperiodic SFTs.



Generalized Baumslag-Solitar groups

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- Combinations: $\langle a, b, c \mid b^{-1}a^3 b = a^5, a^4 c^2 \rangle$
- And more!

Classification by QI

Theorem (Whyte '04)

For any Generalized Baumslag-Solitar group G exactly one of the following is true:

1. $G = BS(1, n)$ for some $n > 1$,
2. G contains a finite index subgroup isomorphic to $\mathbb{F}_n \times \mathbb{Z}$,
3. G is quasi-isometric to $BS(2, 3)$.

Puzzle pieces

Theorem (Aubrun, Schraudner '20, Esnay, Moutot '22)

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The Baumslag-Solitar group $BS(2,3)$ admits a strongly aperiodic SFT.

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1. Flow shift: We create an SFT which specifies an infinite path.
2. Folding: We "fold" an aperiodic configuration (from $\mathbb{Z}^2$ or the hyperbolic plane) along the path specified by the flow.

$\mathbb{F}_n \times \mathbb{Z}$: Flow shift

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$$y_g = s \implies \begin{cases} y_{gs} \neq s^{-1} \\ y_{gs'} = (s')^{-1} \; \forall s' \neq s \\ y_{gt} = s \end{cases}$$

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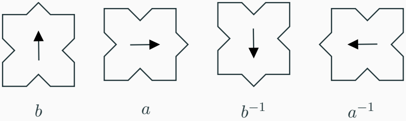

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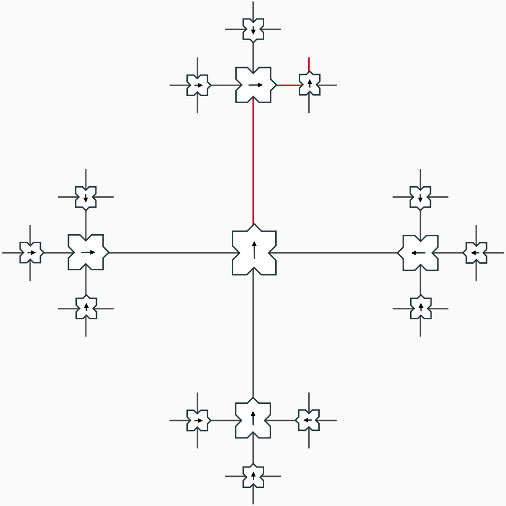


$\mathbb{F}_n \times \mathbb{Z}$: Flow shift

There is a bijection $W : Y_f \to A^{\mathbb{N}}$, that is, every config is determined by a unique infinite word.

Lemma

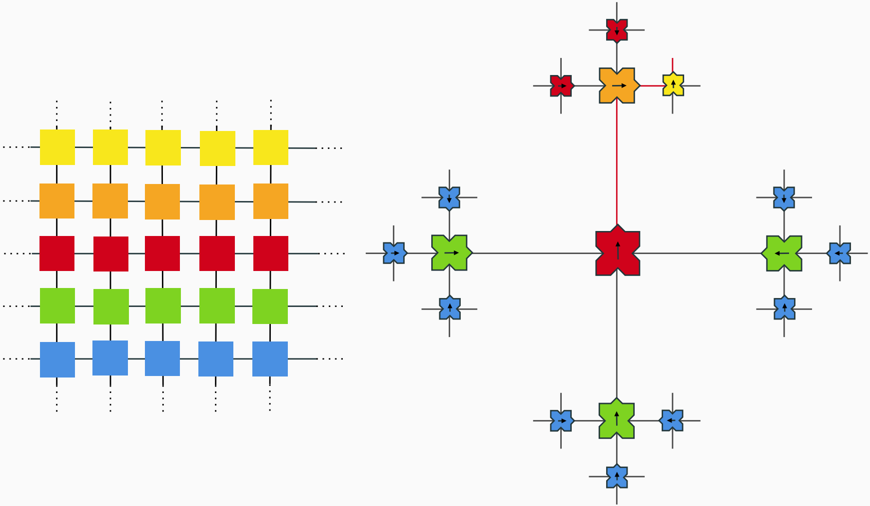
If $y \in Y_f$ has period $gt^k \in \mathbb{F}_n \times \mathbb{Z}$, then $W(y)$ is either the infinite word $g^{\mathbb{N}}$ or the infinite word $(g^{-1})^{\mathbb{N}}$.

$\mathbb{F}_n \times \mathbb{Z}$: Path-folding

Let X be a horizontally expanding nearest neighbor $\mathbb{Z}^2$-SFT. We define a subshift $Z \subseteq X \times Y_f$:

- Horizontal rules rest the same along t.
- Vertical rules follow the direction of the generator at the second coordinate.

$\mathbb{F}_n \times \mathbb{Z}$: Path-folding



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Proposition

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Proposition

There exists a minimal strongly aperiodic $\mathbb{Z}^2$-SFT.

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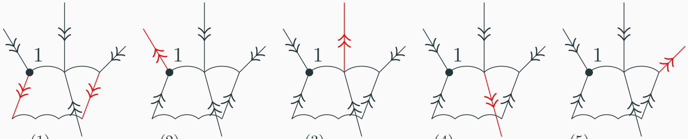
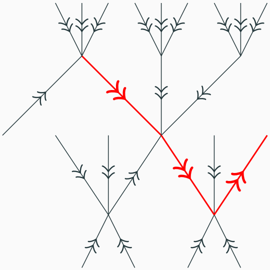
- Local rules:
  - $y_g = y_{ga^2}$ if $y_g \in \{t, at\}$,
  - $y_g = y_{ga^3}$ if $y_g \in \{t^{-1}, at^{-1}, a^2t^{-1}\}$,
  - if $y_g = u$, then $\forall v \in A \setminus \{u\}$: $y_{gv} = v^{-1}$

$BS(2,3)$: Flow shift



$p_1^{(1)} = t$ $p_1^{(2)} = t^{-1}$ $p_1^{(3)} = at$ $p_1^{(4)} = at^{-1}$ $p_1^{(5)} = a^2t^{-1}$



$BS(2,3)$: Periods

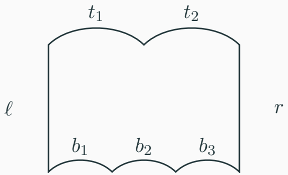
We have a bijection $W : Y_f \to A^{\mathbb{N}}$.

Proposition

If $y \in Y_{\text{flow}}$ has period $g \in BS(2,3)$ with decomposition $g^{-1} = wa^k$, then $W(y)$ is either the infinite word $w^{\mathbb{N}}$ or the infinite word $(w^{-1})^{\mathbb{N}}$.

$BS(2,3)$: Tiles

Wang tiles on $BS(2,3)$ are interpreted as a $7-$tuple:



We say the tile computes a function $f : I \subset \mathbb{R} \to I$ if

$$f\left(\frac{t_1 + t_2}{2}\right) + \ell = \frac{b_1 + b_2 + b_3}{3} + r.$$

$BS(2,3)$: Encoding orbits

Once again we use the function:

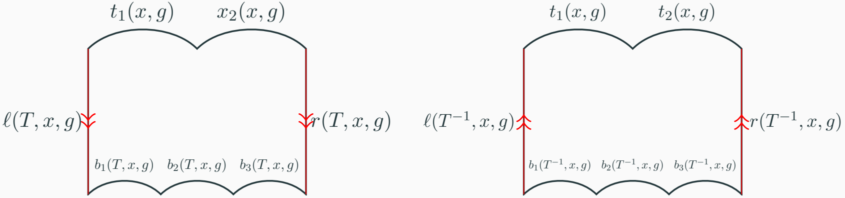
$$T : x \mapsto \begin{cases} \frac{5}{2}x \text{ if } x \in [\frac{1}{10}; 1] \\ \\ \frac{1}{10}x \text{ if } x \in]1; \frac{5}{2}[\end{cases}$$

and its inverse

$$T^{-1} : x \mapsto \begin{cases} 10x \text{ if } x \in]\frac{1}{10}; \frac{1}{4}[\\ \\ \frac{2}{5}x \text{ if } x \in [\frac{1}{4}; \frac{5}{2}] \end{cases}$$

$BS(2,3)$: Path-folding

We define tiles that compute the functions:



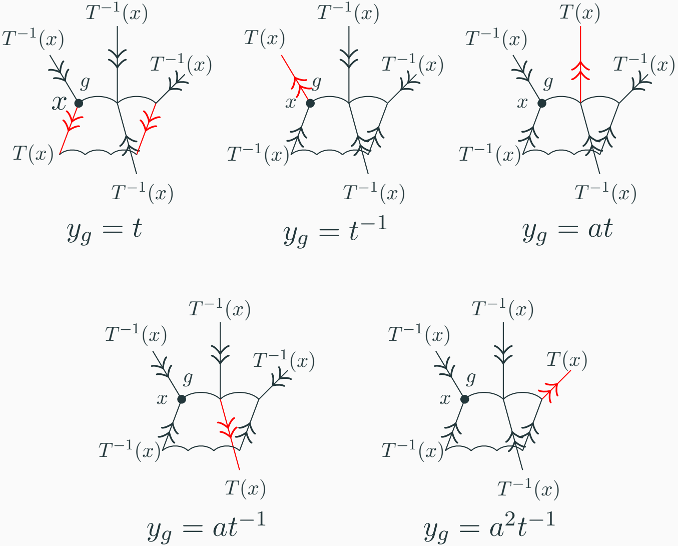
We obtain two finite tileset τ_T and $\tau_{T^{-1}}$.

$BS(2,3)$: Path-folding

We combine the tilesets $\tau = \tau_T \cup \tau_{T^{-1}}$.
The SFT $Z \subseteq \tau^{BS(2,3)} \times Y_f$ is defined as follows:

- if $y_g = t$ then $\tau_g \in \tau_T$;
- if $y_g \in \{at, t^{-1}, t^{-1}, at^{-1}, a^2t^{-1}\}$ then $\tau_g \in \tau_{T^{-1}}$.

$BS(2,3)$: Path-folding



$BS(2,3)$: Main result

Theorem

Z is a strongly aperiodic $BS(2,3)$-SFT.

Sketch:

- If we have a period $g = wa^k$, we have $W(y) = w^{\mathbb{N}}$

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- Then following the flow we find a period for T.

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- Each $\langle a \rangle$-coset contains a k-periodic word
- Then following the flow we find a period for T.
- Therefore, $k = 0$.

Consequences

Adding Cohen's theorem to the mix:

Corollary

Non-residually finite Baumslag-Solitar groups $BS(m, n)$ with $m, n > 1$ and $m \neq n$ admit strongly aperiodic SFTs.

Thank you for listening!

Appendix


APPENDIX
Terre saine
Poitou-Charentes
Votre commune sans pesticides
COMITÉ NATIONAL
Ville Fleurie
POUR LE FLEURISSEMENT DE LA FRANCE
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Graph of groups

Definition

A graph of groups $(\Gamma, \mathcal{G})$ is a connected graph Γ, with a collection $\mathcal{G}$ that includes:

- a vertex group G_v for each $v \in V_\Gamma$,
- an edge group G_e for each $e \in E_\Gamma$, where $G_e = G_{\bar{e}}$,
- a set of injections $\{\alpha_e : G_e \to G_{\mathfrak{t}(e)} \mid e \in E_\Gamma\}$, where $\mathfrak{t}(e)$ is the terminal vertex of e.

If we take all $G_v = G_e = \mathbb{Z}$ we get a Generalized Baumslag-Solitar group.

Graph of groups

Theorem

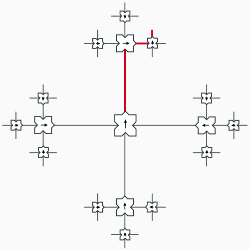
Let $T \subseteq \Gamma$ be a spanning tree. The group $\pi_1(\Gamma, \mathcal{G}, T)$ is isomorphic to a quotient of the free product of the vertex groups, with the free group on the set E_Γ of oriented edges. That is,

$$\left(\mathop{\ast}_{v \in V_\Gamma} G_v * F(E_\Gamma) \right) /R,$$

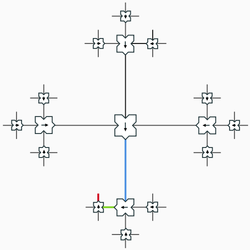
where R is the normal closure of the subgroup generated by the following relations

- $\alpha_{\bar{e}}(h)e = e\alpha_e(h)$, where e is an oriented edge of Γ, $h \in G_e$,
- $\bar{e} = e^{-1}$, where e is an oriented edge of E_Γ,
- $e = 1$ if e is an oriented edge of T_0.

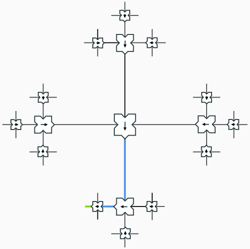
$\mathbb{F}_n \times \mathbb{Z}$: Minimality



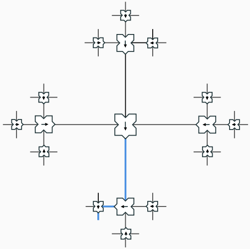
y



$\sigma^{ab}(y)$



$\sigma^{a^2b}(y)$



$\sigma^{(ab)^2}(y)$

Domino Problem

Proposition

All non-$\mathbb{Z}$ Generalized Baumslag-Solitar groups have undecidable domino problem.

Corollary

All non-free Artin groups have undecidable domino problem.