Domino Snake Problems on Groups

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September 19, 2023

FCT 2023

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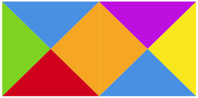
Wang Tiles

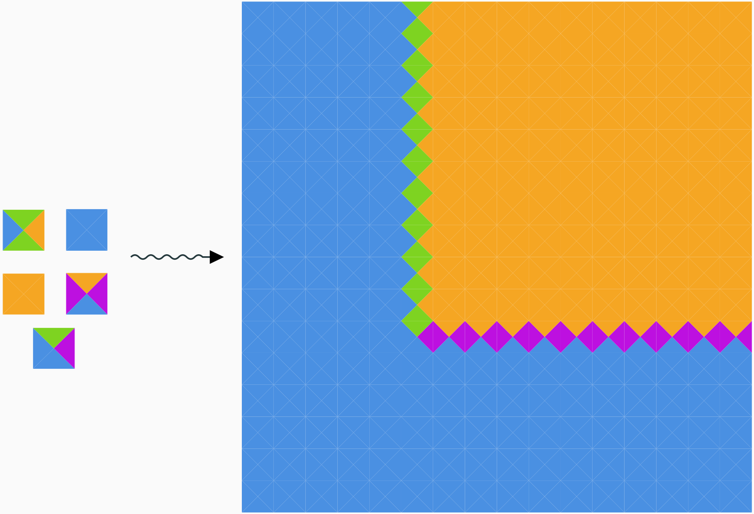
- Our story begins with Wang tiles:

First introduced by Hao Wang in 1961.

- They can be placed side by side if they share the same color along their common border.



Tilings



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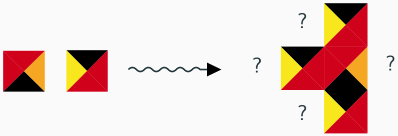
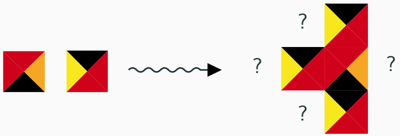


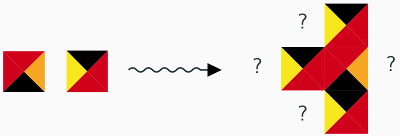
Tiling?



Domino Problem

Given a finite set τ of Wang tiles, does τ tile the plane?

Tiling?



Domino Problem

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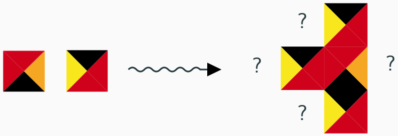
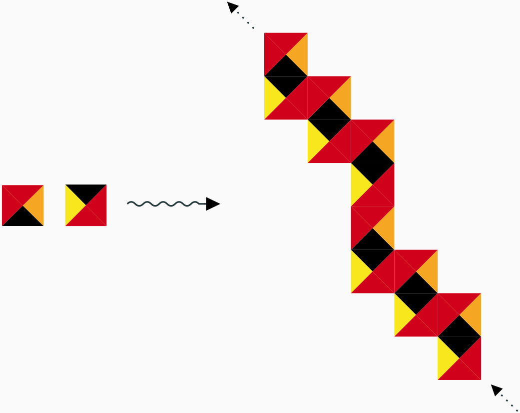
Theorem (Berger '64)

The Domino Problem is undecidable (Π_1^0-comp.).

Snakes?



Snakes!



New Problem

An *a priori* weaker version of the Domino Problem:

Infinite Snake

Given a finite Wang tileset τ, does there exist a snake tiled by τ?

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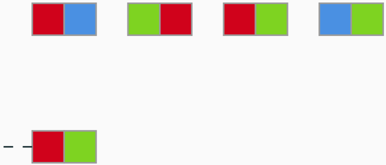
Theorem (Adleman, J. Kari, L. Kari, Reishus '02)

The infinite snake problem in $\mathbb{Z}^2$ is undecidable (Π_1^0-comp).

Lineland



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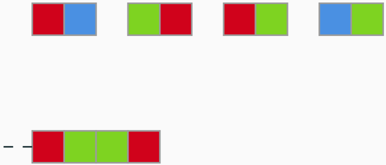

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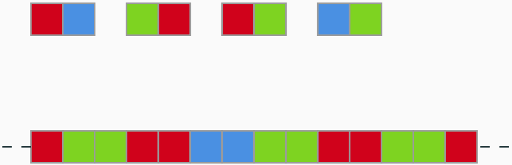

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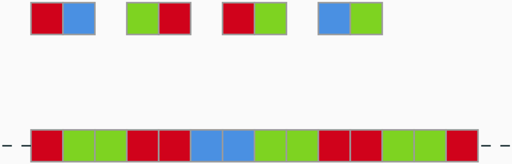
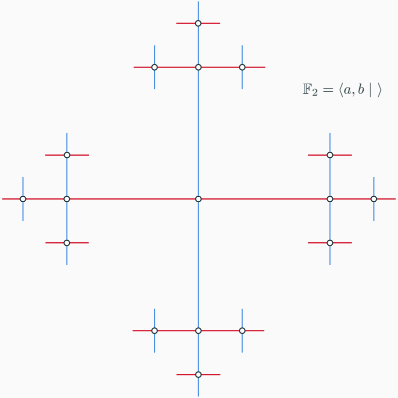


In $\mathbb{Z}$ both the Domino Problem and Infinite Snake Problem are decidable. Why?

Where is the Undecidability?

As was done for the Domino Problem, we study our problem in a particular class of graphs: Cayley graphs of finitely generated infinite groups.

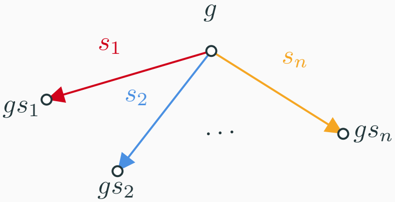


These graphs are infinite, locally finite, regular, transitive, edge labelled.

Groups as Graphs

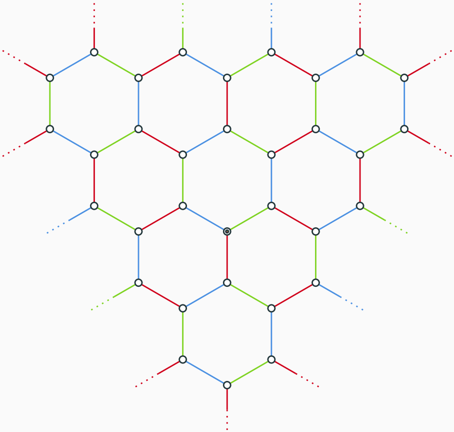
A Cayley graph is defined from a group G along with a finite generating set S:

- Vertices are elements of G,
- There is an edge from g to h if $h = gs^{\pm 1}$.

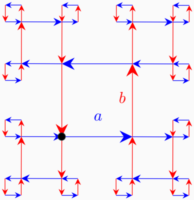


Examples

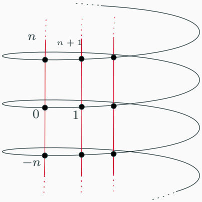
$$\langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^3, (ac)^3 \rangle$$



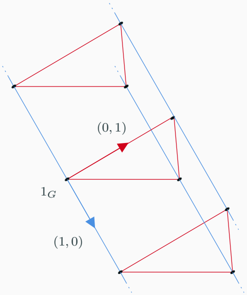
More Examples



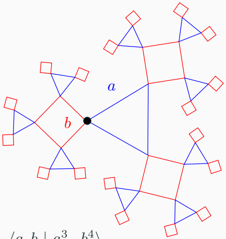
$\langle a, b \mid (ab)^2 \rangle$



$(\mathbb{Z},\ \{1, n\})$



$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

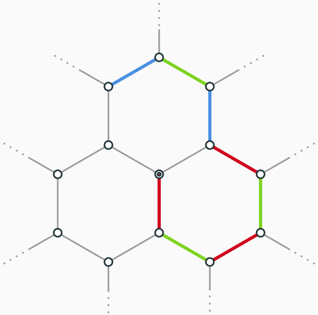


$\langle a, b \mid a^3,\ b^4 \rangle$

Words and Paths

Given a Cayley graph for a group G with generating set S, there is a correspondence between paths and words in $(S \cup S^{-1})^*$.

$ababacbc \longrightarrow$



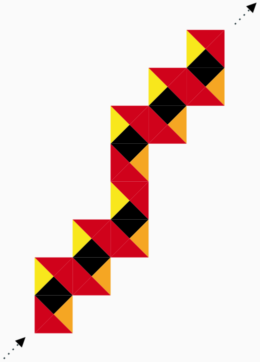
For instance, cycles are described by the set

$$\mathsf{WP}(G, S) = \{w \in (S \cup S^{-1})^* \colon\; w =_G \varepsilon\}.$$

Formal snakes

Let τ be a Wang tileset. A τ-snake is a pair of functions (ω, ζ):

- (Skeleton) $\omega : \mathbb{Z} \to G$ injective s.t. $\omega(i+1)\omega(i)^{-1} \in S \cup S^{-1}$,
- (Scales) $\zeta : \mathbb{Z} \to \tau$ respecting local adjacency rules.



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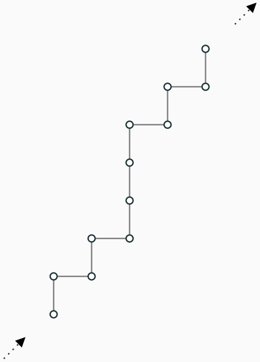

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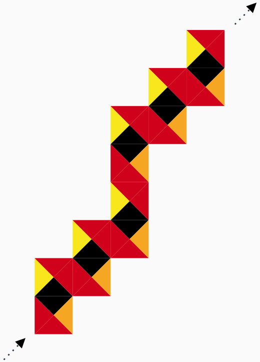

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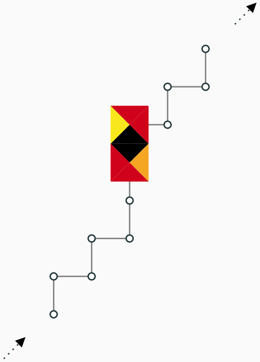
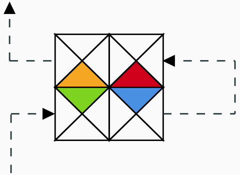


Strong snakes

In particular, we do not care if adjacent, but not sequentially adjacent, tiles match.



Results

Our investigation went along the following lines:

1. **Reducing the problem from one group to another**:
   Virtually nilpotent groups that are not virtually $\mathbb{Z}$ admit a Cayley graph with undecidable infinite snake problem.

2. **Expressing the problem in MSO logic**:
   Virtually free groups have decidable infinite snake problem on all Cayley graphs.

3. **Finding decidability by restricting possible skeletons**:
   The infinite snake problem on $\mathbb{Z}^2$ becomes decidable when considering geodesic skeletons.

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MSO logic and Trees

$$\text{Virtually free} \iff \text{Finite tree-width} \iff \text{Decidable MSO logic}$$

[Muller, Schupp '85] [Kuskey, Lohrey '05]

Therefore, if the infinite snake problem can be expressed in MSO logic, it is decidable for this class of groups.

Theorem

The infinite snake problem is decidable on virtually free groups, for every generating set.

MSO logic and Trees

MSO logic of an S labelled graph Γ consists in:

- Variables are subsets of vertices, along with the constant set $\{v_0\}$,
- an operation $P \cdot s$, representing all the vertices reached from P when reading s,
- Boolean operations $\vee, \wedge, \subseteq, \neg, \ldots$ and quantifiers $\forall, \exists$

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Skeletons

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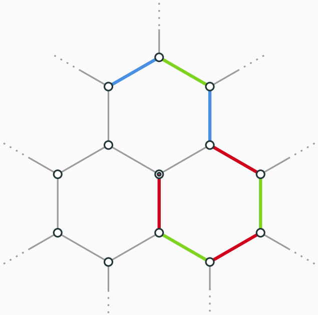
$ababacbc$ ⟶ 

For instance, cycles are described by the set

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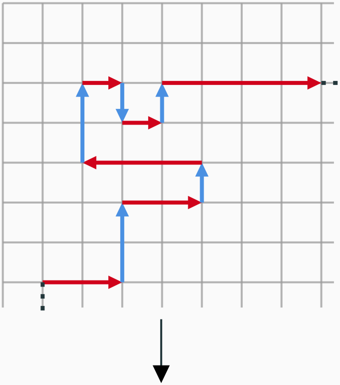
Skeletons

This allows us to understand skeletons as bi-infinite words without loops.



$\ldots a\ a\ b\ b\ a\ a\ b\ a^{-1}\ a^{-1}\ a^{-1}\ b\ b\ a\ b^{-1}\ a\ b\ a\ a\ a\ a \ldots$

Skeletons

Let G be a f.g. group with S a set of generators.

The skeleton set of the pair (G, S) is

$$X_{G,S} = \{x \in (S \cup S^{-1})^{\mathbb{Z}} \mid \forall w \sqsubseteq x,\ w \notin \mathsf{WP}(G,S)\},$$

For instance:

$$X_{\mathbb{Z}^2,\{a,b\}} = \left\{x \in \{a^{\pm 1}, b^{\pm 1}\}^{\mathbb{Z}} \colon\ \forall w \sqsubseteq x, |w|_a \neq |w|_{a^{-1}} \vee |w|_b \neq |w|_{b^{-1}}\right\}.$$

Y Skeletons?

Let $Y \subseteq X_{G,S}$ be a subset of skeletons.

Y-Snake Problem

Given a Wang tileset τ, does there exist a snake tiled by τ whose skeleton is contained in Y?

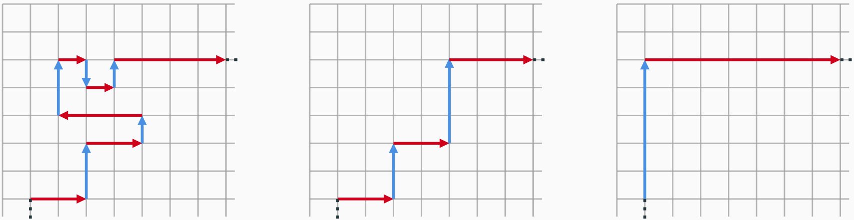
Lemma

If $\mathcal{L}(Y) = \{w \in (S \cup S^{-1})^* \mid \exists x \in Y,\ w \sqsubseteq x\}$ is regular, then the Y-snake problem is decidable.

Geodesic Skeletons

A skeleton of particular interest is the set of bi-infinite geodesics:

$$X^g_{G,S} = \{x \in X_{G,S} \colon\ \forall w \sqsubseteq x, w' =_G w :\ |w| \le |w'|\} \subseteq X_{G,S}$$



$\times$ $\checkmark$ $\checkmark$

$X^g_{\mathbb{Z}^2,\{a,b\}}$ has regular language!

No Curves

Theorem

The Y-snake problem on $\mathbb{Z}^2$ is decidable for:

- Geodesic skeletons,
- 3 or less canonical directions,

or in general:

Theorem

If a group G with generating set S has a set of geodesics that is regular, then its geodesic snake problem is decidable.

This happens with many classes, such as hyperbolic groups.

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Thank you for listening!

Spooky appendix

Groups, Graphs and Geometry

Let $\mathcal{P}$ be a group property (abelian, nilpotent, free, etc). We say a group is virtually $\mathcal{P}$ if it contains a finite index subgroup satisfying $\mathcal{P}$.

For instance, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is virtually $\mathbb{Z}$, the Honeycomb group ($\mathbb{Z}^2 \rtimes S_3$) is virtually nilpotent, $SL(2, \mathbb{Z})$ is virtually free.

$$\text{Virtually nilpotent} \iff \text{Polynomial growth rate,}$$

[Gromov '81]

$$\text{Virtually free} \iff \text{Finite tree-width.}$$

[Muller, Schupp '85]

Very Undecidable Skeletons

Corollary

If Y is an effective $\mathbb{Z}$-subshift, then the Y-skeleton snake problem is Π^0_1.

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What happens if Y is not closed?

Very Undecidable Skeletons

Corollary

If Y is an effective $\mathbb{Z}$-subshift, then the Y-skeleton snake problem is Π_1^0.

What happens if Y is not closed?

Theorem (Ebbinghaus '86)

Let us have the $\mathbb{Z}^2$-skeletal subset

$$Y = \{x \in \{a, b, a^{-1}, b^{-1}\}^{\mathbb{Z}} \mid x \text{ is not eventually a line.}\}$$

Then the Y-skeleton snake problem is Σ_1^1-complete.

Snakes in Space!

What happens in higher dimensions?

Proposition

The infinite snake problem in $\mathbb{Z}^d$, with $d \geq 2$ is undecidable for all generating sets.