Distortion in Automorphisms of Expansive Systems

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Abstract In this work we study the role distortion plays on automorphisms groups of expansive dynamical systems. We begin by generalizing results from subshifts, linking distortion and non-expansivity, to arbitrary expansive systems, and explore the subset of symmetrically distorted automorphisms. Due to the generalization, we are able to determine that expansive automorphisms can never be distorted.

1 Introduction

In symbolic dynamics and group theory, distortion generally refers to an object that grows or moves sub-linearly. In particular, we say that a cellular automaton or an endomorphism acting on a symbolic space is *range distorted* if the local radius of the iterated applications grows at the aforementioned rate. Examples of this behavior can be constructed from Turing machines when viewed as endomorphisms or automorphisms of a symbolic space. Indeed, Guillon and Salo showed in [7] that any aperiodic Turing machine is range distorted. Also, using the so-called conveyor belt technique one can show that on any sofic shift one can define a non-trivial range distorted automorphism.

Analogously, an element of a finitely generated group is said to be *distorted* if its minimal expression on the generating set grows sub-linearly with successive iterations of the element. The two mentioned concepts are related: a group distorted

All authors thank ANID Grant BASAL FB210005 and ACE210010.

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A. Adamatzky (ed.), *Automata and Complexity*, Emergence, Complexity and Computation 42, https://doi.org/10.1007/978-3-030-92551-2_3

automorphism is always range distorted (here we are considering the group of automorphisms). This prompts the fundamental question: does the converse hold?

To address this question, several notions of discrete Lyapunov exponents have been introduced. These objects are closely related to range distortion due to the fact that they quantify the average rate at which any kind of endomorphism on a symbolic space moves information. One recent example are the exponents introduced by Cyr, Franks and Kra in [6], that quantify the rate at which information is moved asymptotically. The novelty of these exponents is that they relate distortion to geometrical properties of the space-time defined by endomorphisms.

In this article, we will show that the phenomenon of distortion is not exclusive to the realm of symbolic systems. This is achieved through the use of M. Boyle and D. Lind's work on expansive dynamical systems [2]. We generalize both the notions of local radius and asymptotic Lyapunov exponents to endomorphisms of arbitrary expansive dynamical systems. Furthermore, the connections between distortion and geometry are preserved. We exemplify this generalization by considering automorphisms of the n-torus.

There is a second question we want to address in this article. Since Boyle and Lind introduced the notions of expansive and non-expansive directions for the study of directional dynamics of an action, there has been one persistent question: which sets can occur as sets of non-expansive directions?

In [2] they showed that this set is closed and, if the domain is infinite, non-empty. Furthermore, they showed that any closed set of directions, with two or more elements is the set of non-expansive directions for some action. Later, Hochman showed in [9] that for every direction, there exists an automorphism of a subshift such that its unique non-expansive direction is the selected one, effectively solving the realization problem. Nevertheless, the subshift built to achieve this result lacks of many natural dynamical properties one would like to get, as transitivity or minimality. This motives the author to ask the following, still open, question: Does any closed non-empty set of directions arise as the set of non-expansive directions of a $\mathbb{Z}^2$-action that is transitive or minimal?

We begin by introducing the necessary concepts from the field of symbolic dynamics from the theory of expansive dynamical systems. We then proceed to generalize the concept of radius to the context of expansive systems, introducing the concept of distorted automorphism. Next, we introduce alternative notions of distortion through the generalization of discrete Lyapunov exponent to the realm of expansive systems. This allows us to establish a connection between non-expansive directions and the asymptotic behavior of an automorphism. In addition, we establish that no expansive automorphism can be distorted. We continue by addressing the question of group distortion in relation to range distortion by studying the set of distorted automorphisms with distorted inverse. Finally, we look at examples of distorted automorphisms, first in the context of subshifts through the use of Turing Machines and then in a non-symbolic example through the study of automorphisms of the torus.

2 Definitions

2.1 Symbolic Spaces

Let Σ be a finite set, which we will henceforth call the alphabet. A *full-shift* is the dynamical system given by the space $\Sigma^{\mathbb{Z}}$, endowed with the product topology, and the shift function $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ given by $\sigma(x)_i = x_{i+1}$, for all $x = (x_i)_{i\in\mathbb{Z}}$ and $i \in \mathbb{Z}$. A σ-invariant and closed subset X of a full-shift $\Sigma^{\mathbb{Z}}$ is called a subshift. Usually we denote a subshift by (X, σ) or X indistinctly.

Given a configuration $x \in X$, for $i < j$ we denote the finite word composed by the symbols in x from index i up to j, $x_i x_{i+1} \dots x_{j-1} x_j$, by $x_{[i,j]}$.

The set of finite words appearing in points or configurations of $\Sigma^{\mathbb{Z}}$ is denoted by Σ^*. Define the language of a subshift $X \subseteq \Sigma^{\mathbb{Z}}$, as the set of all finite words that appear on configurations in X,

$$\mathcal{L}(X) = \{w \in \Sigma^* : w = x_{[n,m]},\ \text{for some integers } n \leq m \text{ and } x \in X\}$$

Definition 1 A subshift $X \subseteq \Sigma^{\mathbb{Z}}$ is said to be a shift of finite type (SFT), if there exists a finite set of (forbidden) words $\mathcal{F} \subseteq \Sigma^*$ such that $x \in X$ if and only if no word in $\mathcal{F}$ appears as a subword of x.

A subshift $X \subseteq \Sigma^{\mathbb{Z}}$ is said to be mixing if there exists $N \in \mathbb{N}$ such that for every pair of words $u, v \in \mathcal{L}(X)$ and $n \geq N$ there exists $w \in \mathcal{L}(X)$ such that $|w| \geq n$ and $uwv \in \mathcal{L}(X)$.

2.2 Endomorphisms of a Shift Space

A map $f\colon X \to Y$ between two subshifts X and Y is called a *morphism* if it is continuous and shift commuting, that is, $f \circ \sigma = \sigma \circ f$. One says the function is an *endomorphism* if $X = Y$ and an *automorphism* if it is also bijective. We will denote the set of all automorphisms of a subshift X by $\mathrm{Aut}(X)$.

Due to the Curtis–Hedlund–Lyndon Theorem [8] we know that every endomorphism $\phi\colon X \to X$, with $X \subseteq \Sigma^{\mathbb{Z}}$, is determined by a local function $\Phi\colon \Sigma^{2N+1} \to \Sigma$ such that $\phi(x)_i = \Phi(x_{[-N+i,N+i]})$. In other words, for two configurations $x, y \in X$,

$$x_{[-N,N]} = y_{[-N,N]} \implies \phi(x)_0 = \phi(y)_0.$$

The minimum $N \in \mathbb{N}$ such that the previous property is satisfied is called the *range* of ϕ, and is denoted by $\mathrm{range}(\phi)$.

We can further classify automorphisms according to how they act on specific configurations.

Definition 2 Let (X, σ) be a subshift and $\phi \in \text{Aut}(X)$. Given a configuration $x \in X$, we say ϕ is *weakly periodic* on x if there exists $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ such that $\phi^p(x) = \sigma^q(x)$.

If ϕ is not weakly periodic for any configuration, then we say it is *aperiodic*.

2.3 Algebraic Distortion

Let us briefly introduce the classical notion of distortion in group theory.

Definition 3 Let G be a finitely generated group and $S \subseteq G$ a symmetric generating set. Given $g \in G$, we define the length of g with respect to S, $\ell_S(g)$, as the smallest non-negative integer n such that g can be written as a product of n elements of S. We write,

$$\ell_S(g) = n.$$

By convention, we use that $\ell_S(e) = 0$.

We note that the function ℓ_S depends on the generating set S only up to a multiplicative constant.

Lemma 1 ([4], Lemma 2.4) *If S_1 and S_2 are two generating sets of G, then there exists a constant $c \geq 1$ such that*

$$\frac{1}{c}\ell_{S_2}(g) \leq \ell_{S_1}(g) \leq c\ell_{S_2}(g),\ \forall g \in G.$$

Definition 4 Let G be a finitely generated group and S a symmetric generating set. The translation length of an element $g \in G$ is defined as the limit:

$$\|g\|_S := \lim_{n\to\infty} \frac{\ell_S(g^n)}{n}.$$

We say g is a distorted element if $\|g\|_S = 0$.

Remark 1 It is important to note that due to Lemma 1, the property of being distorted is independent of the generating set.

2.4 Expansive Dynamical Systems

Let (X, ρ) be a compact metric space with metric ρ, which we assume to be infinite. A $\mathbb{Z}^d$-action Ψ on X is a homomorphism from the additive group $\mathbb{Z}^d$ to the group $\text{Homeo}(X)$ (homeomorphisms of X with composition). Given a subset $F \subseteq \mathbb{R}^d$ we define:

$$\rho_\Psi^F(x, y) = \sup\{\rho(\Psi^n(x), \Psi^n(y)) : n \in F \cap \mathbb{Z}^d\},$$

where Ψ^n is the element in Homeo(X) associated to $n \in \mathbb{Z}^d$. If $F \cap \mathbb{Z}^d = \emptyset$, we write $\rho_\Psi^F(x, y) = 0$.

In this context, an *automorphism* $\phi\colon X \to X$ *for the action of* Ψ is a bi-continuous function such that it commutes with the $\mathbb{Z}^d$-action, that is, $\phi \circ \Psi^n = \Psi^n \circ \phi$, $\forall n \in \mathbb{Z}^d$. The group of all automorphisms for the action of Ψ will be denoted by $\mathrm{Aut}(X, \Psi)$. In the sequel, if there is no ambiguity on the action Ψ, we will just speak about automorphisms of X.

Definition 5 A $\mathbb{Z}^d$-action Ψ on X is *expansive* if there exists $c > 0$ such that

$$\rho_\Psi^{\mathbb{R}^d}(x, y) \leq c \implies x = y.$$

In such a case, c is called the expansivity constant of Ψ.

When $d = 1$, we say Ψ is *positively expansive* if there exists $c > 0$ such that

$$\rho_\Psi^{\mathbb{R}_{0,+}}(x, y) \leq c \implies x = y.$$

For a subset $F \subseteq \mathbb{R}^d$ and $v \in \mathbb{R}^d$, we define

$$\mathrm{dist}(v, F) = \inf\{\|v - w\| : w \in F\},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. For $t > 0$ we define the *thickening of* F *by* t as $F^t = \{v \in \mathbb{R}^d : \mathrm{dist}(v, F) \leq t\}$.

Definition 6 Let Ψ be a $\mathbb{Z}^d$-action on X and $F \subseteq \mathbb{R}^d$. Then, F is expansive for Ψ if there exists $\varepsilon > 0$ and $t > 0$ such that

$$\rho_\Psi^{F^t}(x, y) \leq \varepsilon \implies x = y.$$

If F does not satisfy this condition, it is said to be non-expansive.

When a $\mathbb{Z}^d$ action Ψ is expansive, the following lemma allows us to consider a uniform ε in Definition 6.

Lemma 2 ([2], Lemma 2.3) *Let* Ψ *be an expansive* $\mathbb{Z}^d$*-action on* X*, with expansivity constant* c*. Then, for each expansive subset* $F \subseteq \mathbb{R}^d$ *for* Ψ *there exists* $s > 0$ *such that*

$$\rho_\Psi^{F^s}(x, y) \leq c \implies x = y.$$

3 Generalizing the Range

We are interested in defining a notion of distortion for an automorphism on an arbitrary expansive system. With this in mind, we present the following general lemma.

Lemma 3 *Let Ψ be an expansive $\mathbb{Z}^d$-action on X, with expansivity constant c. Then, for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $\forall x, y \in X$,*

$$\rho_{\Psi}^{B_\infty(0,M)}(x, y) \leq c \implies \rho(x, y) \leq \varepsilon,$$

where $B_\infty(0, M) = \{v \in \mathbb{R}^d : ||v||_\infty = \max_{1\leq i\leq d} |v_i| \leq M\}$.

Proof We proceed by contradiction. Let ε be such that for all $m \in \mathbb{N}$, there are $x_m, y_m \in X$ such that $\rho_{\Psi}^{B_\infty(0,m)}(x_m, y_m) \leq c$ and $\rho(x_m, y_m) > \varepsilon$. Since X is compact, we have a subsequence $(m_i)_{i\in\mathbb{N}}$ such that the sequences $(x_{m_i})_{i\in\mathbb{N}}$ and $(y_{m_i})_{i\in\mathbb{N}}$ converge to $\bar{x}$ and $\bar{y}$ respectively.

Let us consider $\eta > 0, n \in \mathbb{Z}^d$ and $I \in \mathbb{N}$ such that $\forall i \geq I, m_i \geq \max\{\|n\|_\infty, m_I\}$

$$\rho(\Psi^n x_{m_i}, \Psi^n \bar{x}) \leq \frac{\eta}{2} \text{ and } \rho(\Psi^n y_{m_i}, \Psi^n \bar{y}) \leq \frac{\eta}{2}.$$

Then,

$$\begin{aligned}\rho(\Psi^n \bar{x}, \Psi^n \bar{y}) &\leq \rho(\Psi^n x_{m_i}, \Psi^n \bar{x}) + \rho(\Psi^n x_{m_i}, \Psi^n y_{m_i}) + \rho(\Psi^n y_{m_i}, \Psi^n \bar{y}) \\ &\leq \eta + c.\end{aligned}$$

By taking $\eta \to 0$ (m_i is always greater than $\|n\|_\infty$) we obtain

$$\rho(\Psi^n \bar{x}, \Psi^n \bar{y}) \leq c, \ \forall n \in \mathbb{Z}^d.$$

Thus, using that Ψ is expansive with constant c, we get that $\bar{x} = \bar{y}$. Therefore, for a sufficiently large i

$$\begin{aligned}\rho(x_{m_i}, y_{m_i}) &\leq \rho(x_{m_i}, \bar{x}) + \rho(y_{m_i}, \bar{y}) \\ &\leq \varepsilon,\end{aligned}$$

which is a contradiction.

3.1 Automorphisms on Expansive Systems

From this point onward, we will work in the following context. Let $T : X \to X$ be a homeomorphism on the compact metric space (X, ρ). In this setting we write

Aut(X, T) for the group of automorphisms of X commuting with T. Observe that T defines a $\mathbb{Z}$-action on X, so we can use the previously defined notations. Recall that given a subset $F \subseteq \mathbb{R}$

$$\rho_T^F(x, y) = \sup\{\rho(T^n x, T^n y) : n \in F \cap \mathbb{Z}\},$$

and that the system (X, T) is expansive if there exists a constant $c > 0$ such that

$$\rho_T^{\mathbb{R}}(x, y) \leq c \implies x = y.$$

Definition 7 Let (X, T) be an expansive system of constant $c > 0$ and $\phi \in$ Aut (X, T). We call *range* of ϕ to the minimum $M \in \mathbb{N}$ such that $\forall x, y \in X$,

$$\rho_T^{[-M,M]}(x, y) \leq c \implies \rho(\phi(x), \phi(y)) \leq c,$$

and we denote it by range(ϕ).

It is clear that if (X, T) is a subshift, with T being the shift map, the previous definition coincides with the usual notion of the radius of an automorphism, as this space is expansive of constant $c = 1/2$.

In the general case where (X, T) is an expansive system of constant $c > 0$, the existence of M is ensured by the following. Since $\phi \in$ Aut(X, T) is continuous over the compact set X, then it is uniformly continuous. Therefore, we have that there exists $\delta > 0$ such that for all $x, y \in X$:

$$\rho(x, y) \leq \delta \implies \rho(\phi(x), \phi(y)) \leq c.$$

By applying Lemma 3, we have that there exists $M \in \mathbb{N}$ such that for all $x, y \in X$:

$$\rho_T^{[-M,M]}(x, y) \leq c \implies \rho(x, y) \leq \delta,$$

and therefore,

$$\rho_T^{[-M,M]}(x, y) \leq c \implies \rho(\phi(x), \phi(y)) \leq c.$$

Now consider $\phi \in$ Aut(X, T). With both ϕ and T we can construct a $\mathbb{Z}^2$-action Ψ defined by: if $n = (n_1, n_2) \in \mathbb{Z}^2$ then $\Psi^n = \phi^{n_2} \circ T^{n_1}$. We also denote the dynamical system defined by the action Ψ by (X, T, ϕ).

Definition 8 Let $E, F \subseteq \mathbb{R}^2$, and Ψ an expansive $\mathbb{Z}^2$-action on X. We say E codifies F if for all $v \in \mathbb{R}^2$,

$$\rho_\Psi^{E+v}(x, y) \leq c \implies \rho_\Psi^{F+v}(x, y) \leq c.$$

Using this terminology, the range of $\phi \in \mathrm{Aut}(X, T)$ can be understood as the minimum $M \in \mathbb{N}$ such that $[-M, M] \times \{0\}$ codifies $\{(0, 1)\}$ for the $\mathbb{Z}^2$-action (X, Ψ) induced by T and ϕ as defined above.

Lemma 4 *Let (X, T) be an expansive system of constant $c > 0$, and consider $\phi, \psi \in \mathrm{Aut}(X, T)$. Then,*

$$\mathrm{range}(\phi \circ \psi) \leq \mathrm{range}(\phi) + \mathrm{range}(\psi).$$

In particular, the sequence $(\mathrm{range}(\phi^n))_{n\in\mathbb{N}}$ is subadditive.

Proof Let $M = \mathrm{range}(\phi)$ and $N = \mathrm{range}(\psi)$. If we have $\rho_T^{[-(M+N),M+N]}(x, y) \leq c$, then

$$\forall t \in [-(M + N), M + N] : \ \rho(T^t x, T^t y) \leq c.$$

If we fix $m \in [-M, M]$ and define $\bar{x} = T^m x$, $\bar{y} = T^m y$, from the previous inequality we obtain that:

$$\forall n \in [-N, N] : \ \rho(T^n \bar{x}, T^n \bar{y}) \leq c.$$

By definition of range, this means that $\rho(\psi(\bar{x}), \psi(\bar{y})) \leq c$. Since this is possible for any $m \in [-M, M]$, we have:

$$\forall m \in [-M, M] : \ \rho(T^m \psi(x), T^m \psi(y)) \leq c,$$

which implies that $\rho(\phi \circ \psi(x), \phi \circ \psi(y)) \leq c$, and therefore, $\mathrm{range}(\phi \circ \psi) \leq N + M$.

Since $(\mathrm{range}(\phi^n))_{n\in\mathbb{N}}$ is a subadditive sequence, by Fekete's Lemma, we have that the following definition makes sense.

Definition 9 Let (X, T) be an expansive system of constant $c > 0$. The *asymptotic range* of $\phi \in \mathrm{Aut}(X, T)$ is defined by

$$\mathrm{range}_\infty(\phi) := \lim_{n\to\infty} \frac{\mathrm{range}(\phi^n)}{n}.$$

If $\mathrm{range}_\infty(\phi) = 0$ we say ϕ is *range distorted*, and denote the set of all range distorted automorphisms in $\mathrm{Aut}(X, T)$ by $RD(X, T)$.

The following simple proposition can be deduced from definition and previous lemma.

Proposition 1 *Let $\phi, \psi \in \mathrm{Aut}(X, T)$. We have,*

1. $\mathrm{range}_\infty(\psi \circ \phi \circ \psi^{-1}) = \mathrm{range}_\infty(\phi)$,
2. $\mathrm{range}_\infty(\phi^p) = p \cdot \mathrm{range}_\infty(\phi)$ *for* $p \in \mathbb{N}$,
3. *if ψ and ϕ commute, then* $\mathrm{range}_\infty(\psi \circ \phi) \leq \mathrm{range}_\infty(\psi) + \mathrm{range}_\infty(\phi)$.

4 Alternative Notion of Distortion

The definition of asymptotic range defined in previous section concerns the average evolution of the symmetric window with which an automorphism of an expansive system (X, T) is computed. To complement this analysis, we introduce an alternative notion of distortion through the use of the Lyapunov exponents presented by Cyr et al. in [6]. These exponents serve to study the average speed at which information asymptotically propagates through the automorphism. This notion will later be shown to be very important because of their connection to some kind of geometry associated to the automorphism.

In what follows we fix an expansive dynamical system (X, T) of expansive constant $c > 0$. Recall (X, ρ) is a compact metric space and T is a homeomorphism of X.

Lemma 5 *Let* (X, T) *be an expansive system with expansivity constant* $c > 0$ *and* $\phi \in \mathrm{Aut}(X, T)$*. Then,*

$$\rho_T^{[0,+\infty)}(x, y) \leq c \implies \rho_T^{[\mathrm{range}(\phi),+\infty)}(\phi(x), \phi(y)) \leq c.$$

That is, $[0, +\infty) \times \{0\}$ *codifies* $[\mathrm{range}(\phi), +\infty) \times \{1\}$ *in* (X, T, ϕ).

Proof We begin by simplifying the notation by writing $r = \mathrm{range}(\phi)$. Let $x, y \in X$ be such that $\rho_T^{[0,+\infty)}(x, y) \leq c$. Then, in particular, we have that

$$\rho_T^{[-(r+k),r+k]}(T^{r+k}x, T^{r+k}y) \leq c, \quad \forall k \geq 0.$$

By the definition of range, we have that

$$\rho(\phi(T^{r+k}x), \phi(T^{r+k}y)) \leq c, \quad \forall k \geq 0,$$

which, by taking supremum over k, can be re-written as

$$\rho_T^{[r,+\infty)}(\phi(x), \phi(y)) \leq c.$$

This finishes the proof. □

We recall from the previous section that we can apply the notion of coding to the $\mathbb{Z}^2$-action Ψ defined by the expansive action T and an automorphism $\phi \in \mathrm{Aut}(X, T)$. We consider the following sets:

$$C^-(\phi) = \{k \in \mathbb{Z} : (-\infty, 0] \times \{0\} \text{ codifies } (-\infty, k] \times \{1\}\},$$

$$C^+(\phi) = \{k \in \mathbb{Z} : [0, \infty) \times \{0\} \text{ codifies } [k, \infty) \times \{1\}\}.$$

Due to Lemma 5, both sets are non-empty. This allows us to define the quantities:

$$W^-(n,\phi) = \sup C^-(\phi^n),$$

$$W^+(n,\phi) = \inf C^+(\phi^n).$$

By definition, we have that for $n \geq 1$, $W^{\pm}(n,\phi) = W^{\pm}(1,\phi^n)$. In addition,

Lemma 6 *Let (X,T) be an expansive system and consider $\phi, \psi \in \mathrm{Aut}(X,T)$. Then,*

$$W^+(n,\phi\psi) \leq W^+(n,\phi) + W^+(n,\psi) \text{ and } W^-(n,\phi\psi) \geq W^-(n,\phi) + W^-(n,\psi).$$

In particular, the sequences $(W^+(n,\phi))_{n\in\mathbb{N}}$ and $(-W^-(n,\phi))_{n\in\mathbb{N}}$ are subadditive.

Again, Fekete's Lemma allows us to make the following definition:

Definition 10 ([6], *Definition 3.12*) Let (X,T) be an expansive system. Given $\phi \in \mathrm{Aut}(X,T)$, we define the exponents of Cyr, Franks and Kra by:

$$\alpha^-(\phi) = \lim_{n\to\infty} \frac{W^-(n,\phi)}{n},$$

$$\alpha^+(\phi) = \lim_{n\to\infty} \frac{W^+(n,\phi)}{n}.$$

Definition 11 We say an automorphism $\phi \in \mathrm{Aut}(X,T)$ of an expansive system (X,T) is *Lyapunov distorted* if $\alpha^{\pm}(\phi) = 0$. We denote the set of all Lyapunov distorted automorphisms in $\mathrm{Aut}(X,T)$ by

$$AD(X,T) = \{\phi \in \mathrm{Aut}(X,T) : \alpha^{\pm}(\phi) = 0\}.$$

These exponents satisfy some very useful properties.

Proposition 2 *Let (X,T) be an expansive system and consider $\phi \in \mathrm{Aut}(X,T)$. We have the following properties:*

1. *For all $k \in \mathbb{Z}$, $\alpha^{\pm}(T^k\phi) = \alpha^{\pm}(\phi) + k$.*
2. *For all $m \in \mathbb{N}$, $\alpha^{\pm}(\phi^m) = m\alpha^{\pm}(\phi)$.*
3. *If $\psi \in \mathrm{Aut}(X,T)$ commutes with ϕ, then:*

$$\alpha^+(\phi\psi) \leq \alpha^+(\phi) + \alpha^+(\psi) \text{ and } \alpha^-(\phi\psi) \geq \alpha^-(\phi) + \alpha^-(\psi).$$

4. *$\alpha^+(\phi) + \alpha^+(\phi^{-1}) \geq 0$ and $\alpha^-(\phi) + \alpha^-(\phi^{-1}) \leq 0$.*
5. *If X is an infinite subshift, then $\alpha^-(\phi) \leq \alpha^+(\phi)$.*

Proof The first property follows directly from the fact that $W^{\pm}(n,T^k\phi) = nk + W^{\pm}(n,\phi)$. The second one comes from:

$$\lim_{n\to\infty} \frac{W^{\pm}(nm,\phi)}{n} = m \cdot \lim_{n\to\infty} \frac{W^+(nm,\phi)}{nm}.$$

Next, for property 3, we see that if $\phi, \psi \in \text{Aut}(X, T)$ commute, then

$$W^+(n, \phi\psi) = W^+(1, (\phi\psi)^n) = W^+(1, \phi^n\psi^n) \leq W^+(n, \phi) + W^+(n, \psi).$$

Property 4 follows from property 3 and the fact that $\alpha^{\pm}(\text{id}) = 0$.

The last property was proved in Proposition 3.15 of [6]. □

Using the following lemma we can see that, in the case of subshifts, Lyapunov distortion is weaker than range distortion. Given $\phi \in \text{Aut}(X, T)$ we denote the interval $[-W^+(n, \phi), -W^-(n, \phi)]$ by $I(n, \phi)$.

Lemma 7 *Let (X, T) be an expansive system and consider $\phi \in \text{Aut}(X, T)$. If J is an interval that ϕ^n-codes $\{0\}$, then $I(n, \phi) \subseteq J$.*

Proof Let $J = [a, b]$ be an interval that ϕ^n-codes $\{0\}$. Then, $(-\infty, 0]$ must ϕ^n-code $(-\infty, -b]$ and $[0, \infty)$ must ϕ^n-code $[-a, \infty)$. We conclude by using the definition of $I(n, \phi)$.

□

Lemma 8 *Let (X, T) be an expansive system and consider $\phi \in \text{Aut}(X, T)$. Then,* $\text{range}_\infty(\phi) \geq \max\left\{\alpha^+(\phi), -\alpha^-(\phi)\right\}$.

Proof The result follows from previous lemma by noting that the interval $[-\text{range}(\phi^n), \text{range}(\phi^n)]$ ϕ^n-codes $\{0\}$. □

Proposition 3 *Let (X, σ) be an infinite subshift. Then, $RD(X, \sigma) \subseteq AD(X, \sigma)$.*

Proof For $\phi \in \text{Aut}(X, \sigma)$, due to (5) on Proposition 2 and Lemma 8, we know that

$$\text{range}_\infty(\phi) \geq \alpha^+(\phi) \geq \alpha^-(\phi) \geq -\text{range}_\infty(\phi),$$

which concludes the proof. □

We can see that in the context of SFTs, the two notions are in fact equivalent. To see this, we first need an auxiliary result.

Lemma 9 ([6], Lemma 3.21) *Let (X, σ) be an SFT and $\phi \in \text{Aut}(X, \sigma)$. Then, there is a constant $C(\phi)$ such that*

$$\frac{|I(n, \phi)| - 1}{2} \leq \text{range}(\phi^n) \leq |I(n, \phi)| + C(\phi).$$

If X is a full-shift we can take $C(\phi) = 0$.

Theorem 1 *Let (X, σ) be an SFT. Then, $AD(X, \sigma) = RD(X, \sigma)$.*

Proof By diving by n and taking limit on the expression given by Lemma 9, we conclude. □

5 Geometry and Distortion

In [6], Cyr, Franks and Kra showed that there is a connection between discrete Lyapunov exponents and the geometry of the $\mathbb{Z}^2$-system (X, σ, ϕ), where $\phi \in \mathrm{Aut}(X, \sigma)$ and X is a subshift. This connection was first explored by Hochman in [9] through the notion of prediction shapes. We generalize these result to the context of expansive systems (X, T). Finally, in the context of subshifts, we connect the newly introduced direction exponents to the standard ones. We relate the fact of having these exponents equal to zero to having non-expansive directions.

The following theorems connect the geometry of the space-time of the automorphisms with its asymptotic behavior.

Theorem 2 *Let (X, T) be an expansive system and consider $\phi \in \mathrm{Aut}(X, T)$. Then, the lines defined by $x = \alpha^+(\phi)y$ and $x = \alpha^-(\phi)y$ are not expansive.*

The proof of this fact is very technical and can be retraced step by step from [6].

Proof Let $c > 0$ be the expansive constant of (X, T). We will only look at the case where there exists a constant $D > 0$ such that

$$0 \leq W^+(k, \phi) - k\alpha^+(\phi) < D, \quad \forall k \geq 0.$$

It is possible to see that due to the definitions at play, if we have two elements $x, y \in X$, such that

$$\rho_T^{[0,\infty)}(x, y) \leq c,$$

then for all $j \geq 0$ and $i \geq D + \alpha^+(\phi)j$,

$$\rho(T^i \phi^j x, T^i \phi^j y) \leq c.$$

We see that this can be interpreted as x and y coinciding on the lower half space of the line $i = D + \alpha^+(\phi)j$.

For every n, because of the definition of $W^+(n, \phi)$, we have $x_n, y_n \in X$ such that

$$\rho_T^{[0,\infty)}(x_n, y_n) \leq c,$$

but, $\rho(T^{W^+(n)-1}\phi^n x_n, T^{W^+(n)-1}\phi^n y_n) > c$. By defining $\hat{x}_n := T^{W^+(n)}\phi^n x_n$ and $\hat{y}_n := T^{W^+(n)}\phi^n y_n$, we can see that,

$$\rho(T^{-1}\hat{x}_n, T^{-1}\hat{y}_n) > c,$$

and for all $j \geq -n$ and $i \geq D + \alpha^+(\phi)j$,

$$\rho(T^i \phi^j \hat{x}_n, T^i \phi^j \hat{y}_n) \leq c.$$

Next, we use the compactness of X to find a convergent subsequences, that converge to $\hat{x} = \lim \hat{x}_n$ and $\hat{y} = \lim \hat{y}_n$.

Let us have an arbitrary $\varepsilon > 0$. Then, due to the continuity of T and ϕ, for sufficiently large n,

$$\begin{aligned} c &< \rho(T^{-1}\hat{x}_n, T^{-1}\hat{y}_n), \\ &\leq \rho(T^{-1}\hat{x}_n, T^{-1}\hat{x}) + \rho(T^{-1}\hat{x}, T^{-1}\hat{y}) + \rho(T^{-1}\hat{y}, T^{-1}\hat{y}_n), \\ &\leq 2\varepsilon + \rho(T^{-1}\hat{x}, T^{-1}\hat{y}). \end{aligned}$$

Therefore, $c < \rho(T^{-1}\hat{x}, T^{-1}\hat{y})$. Analogously, for an arbitrary $\varepsilon > 0$, sufficiently large n and (i, j) such that $i > D + \alpha^+(\phi)j$,

$$\begin{aligned} \rho(T^i\phi^j\hat{x}, T^i\phi^j\hat{y}) &\leq \rho(T^i\phi^j\hat{x}_n, T^i\phi^j\hat{x}) + \rho(T^i\phi^j\hat{x}, T^i\phi^j\hat{y}) + \rho(T^i\phi^j\hat{y}, T^i\phi^j\hat{y}_n), \\ &\leq 2\varepsilon + c. \end{aligned}$$

In other words, for $i > D + \alpha^+(\phi)j$,

$$\rho(T^i\phi^j\hat{x}, T^i\phi^j\hat{y}) \leq c.$$

This proves that the line defined by $x = \alpha^+ y$ is not expansive. □

Theorem 3 *Let (X, T) be an expansive dynamical system, $\phi \in \mathrm{Aut}(X, T)$ and L a line in $\mathbb{R}^2$ given by $x = my$. If $m > \max\{\alpha^+(\phi), -\alpha^-(\phi^{-1})\}$ or $m < \min\{\alpha^-(\phi), -\alpha^+(\phi^{-1})\}$, then L is expansive.*

Proof Let us first show that if $m > \alpha^+(\phi)$, then L is left-expansive.

We take $x, y \in X$ such that:

$$\rho(T^n\phi^k(x), T^n\phi^k(y)) \leq c, \quad \forall (n, k) \in \mathbb{Z}^2 \text{ such that } n > mk.$$

Because $m > \alpha^+(\phi)$, the vector defined by $(\alpha^+(\phi), 1)$ is not parallel to L.

By Definition 10, for sufficiently large n, the vector $(W^+(n), n)$ is not parallel to L.

Next, let us have (u_0, v_0), an arbitrary point to the left of L (that is, $u_0 < mv_0$). There exists $n_0 > 0$ such that if $u_1 = u_0 - W^+(n_0)$ and $v_1 = v_0 - n_0$, then (u_1, v_1) is to the right of L. Therefore, the line given by $\{(t, v_1) : u_1 \leq t\}$ is to the right of L and codifies (u_0, v_0) by definition of $W^+(n_0)$. This shows that L is left expansive.

Analogously, if $m < \alpha^-(\phi)$ then L is right expansive.

Lastly, we can see that the transformation $r(x, y) = (x, -y)$ allows us to move between the $\mathbb{Z}^2$-systems (X, T, ϕ) and (X, T, ϕ^{-1}). Consequently, L is right-expansive (left) on the first system if and only if $r(L)$ is left-expansive (right) one the second one. This fact concludes the proof □

By combining these results, we arrive at the fundamental connection between distortion and non-expansive subspaces.

Corollary 1 *Let (X, T) be an expansive system and consider $\phi \in \mathrm{Aut}(X, T)$. Then, $\phi, \phi^{-1} \in AD(X, T)$ if and only if $x = 0$ is the only non-expansive direction of ϕ.*

A first consequence of this connection is the fact that expansive automorphisms can not be Lyapunov distorted.

Theorem 4 *Let (X, T) be an expansive system of constant $c > 0$ and consider an expansive automorphism $\phi \in \mathrm{Aut}(X, T)$ of constant $\delta > 0$. Then, $\phi \notin AD(X)$.*

Proof Let us call Ψ the joint $\mathbb{Z}^2$-action of T and ϕ. Due to Lemma 3 on the expansive system (X, T), we know that there exists $M \in \mathbb{N}$ such that:

$$\forall x, y \in X : \rho_T^{[-M,M]}(x, y) \leq c \implies \rho(x, y) \leq \delta.$$

Now, let us see that L_0, defined by $x = 0$, is an expansive direction. If we have $x, y \in X$ such that

$$\rho_\Psi^{L_0^M}(x, y) \leq c,$$

in particular we have that,

$$\forall n \in \mathbb{Z} : \rho_T^{[-M,M]}(\phi^n(x), \phi^n(y)) \leq c.$$

This implies that,

$$\forall n \in \mathbb{Z} : \rho(\phi^n(x), \phi^n(y)) \leq \delta.$$

Given that ϕ is expansive, this means that $x = y$. We conclude by using Theorem 2. □

6 Symmetric Distortion Subset

Let (X, T) be an expansive system. We denote the subset of range distorted automorphisms of (X, T) with a range distorted inverse by:

$$\mathfrak{D}(X, T) = \{\phi \in \mathrm{Aut}(X, T) : \mathrm{range}_\infty(\phi) = \mathrm{range}_\infty(\phi^{-1}) = 0\},$$

and the subgroup of distorted elements in $\mathrm{Aut}(X, T)$ (in the algebraic sense) by $GD(X, T)$.

Proposition 4 $GD(X, T) \subseteq \mathfrak{D}(X, T)$.

Proof Let ϕ be a distorted element of $\mathrm{Aut}(X, T)$. This means that there exists a finitely generated subgroup G of $\mathrm{Aut}(X, T)$ such that $\|\phi\|_S = 0$, for a symmetric generating set S. Then, by Lemma 4,

$$\mathrm{range}(\phi^n) \leq \ell_S(\phi^n) \cdot \max_{s \in S}\{\mathrm{range}(s)\}.$$

Dividing the expression by n and taking limit, ϕ is range distorted. We conclude by noting that if ϕ is group distorted, its inverse also is. □

For the next Lemma, we recall that a map $f\colon X \to X$ is said to be equicontinuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho(x, y) \leq \delta \implies \rho(f^n(x), f^n(y)) \leq \varepsilon, \quad \forall n \in \mathbb{Z}.$$

Lemma 10 *Let $\phi \in \mathrm{Aut}(X, T)$ be an equicontinuous automorphism of the expansive system (X, T). Then $\phi \in \mathfrak{D}(X, T)$.*

Proof Because ϕ is equicontinuous, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\rho(x, y) \leq \delta \implies \rho(\phi^n(x), \phi^n(y)) \leq \varepsilon, \quad \forall n \in \mathbb{Z}.$$

By picking $\varepsilon = c$, Lemma 3 tells us that there exists $M > 0$ such that

$$\rho_T^{[-M,M]}(x, y) \leq c \implies \rho(x, y) \leq \delta.$$

This implies that,

$$\rho_T^{[-M,M]}(x, y) \leq c \implies \rho(\phi^n(x), \phi^n(y)) \leq c, \quad \forall n \in \mathbb{Z},$$

that is, $\mathrm{range}(\phi^n) \leq M$ for all $n \in \mathbb{Z}$. We conclude that $\mathrm{range}_\infty(\phi) = \mathrm{range}_\infty(\phi^{-1}) = 0$. □

Remark 2 As a consequence of the Arzelà–Ascoli Theorem, any compact subgroup K of $\mathrm{Aut}(X, T)$ satisfies $K \subseteq \mathfrak{D}(X, T)$.

We are interested in understanding the structure of $\mathfrak{D}(X, T)$. In the general setting, this set is not a subgroup of $\mathrm{Aut}(X, T)$, as is shown in Remark 3 below, where we show an automorphism which is not distorted, but is a composition of two distorted automorphisms through an example due to Schmieding [12].

Remark 3 Let $X = \{0, 1, 2\}^{\mathbb{Z}}$ be the full-shift on 3 symbols. Let ϕ_1 be the marker automorphism that permutes 000111 with 002111, and ϕ_2 the marker automorphism that permutes 000111 with 002111 (marker automorphisms are presented in great detail in [3]). If we define $\phi = \phi_2 \circ \phi_1$, it is possible to see ϕ is not distorted even though both ϕ_1 and ϕ_2 are symmetrically distorted.

The next lemma follows directly from Proposition 1.

Lemma 11 *Let (X, T) be an expansive system and consider $\phi, \psi \in \mathfrak{D}(X, T)$. We have the following properties:*

1. *If $[\phi, \psi] = \mathrm{id}$, then $\phi \circ \psi \in \mathfrak{D}(X, T)$, where $[\phi, \psi] = \phi\psi\phi^{-1}\psi^{-1}$.*
2. *For all $\varphi \in \mathrm{Aut}(X, T)$, $\varphi \circ \phi \circ \varphi^{-1} \in \mathfrak{D}(X, T)$.*

3. *$\phi^p \in \mathfrak{D}(X, T)$, for all $p \in \mathbb{N}$.*

Proposition 5 *Let (X, σ) be a mixing SFT. Then, $\mathfrak{D}(X, \sigma)$ contains an isomorphic copy of every finite group.*

Proof This result follows from the fact that every finite order automorphism is equicontinuous and the Kim and Roush Theorem [10], that states that the automorphism group of a mixing SFT contains an isomorphic copy of the automorphisms group of any n-full shift for $n \geq 2$. □

Finally, let us generalize the fact that the subgroup generated by the action T has a trivial intersection with $\mathfrak{D}(X, T)$.

Lemma 12 *Let (X, T) be an expansive system of expansive constant $c > 0$. If X is infinite, then* $\text{range}_\infty(T) = 1$.

To prove this result we make use of a following result by Schwartzman about infinite systems.

Theorem 5 ([2], Theorem 3.9) *Let T be a homeomorphism of an infinite compact metric space (X, ρ) and $\delta > 0$. Then, there exists two distinct $x, y \in X$ such that $\rho(T^n x, T^n y) \leq \delta$ for all $n \geq 0$.*

Proof (of Lemma 12) It is evident that $\text{range}(T^n) \leq n$. To obtain the other bound, by applying Theorem 5 to T^{-1}, we obtain two distinct points $x, y \in X$ such that $\rho_T^{(-\infty,0]}(x, y) \leq c$. Given that T is expansive and the points are different, we choose the smallest $m > 0$ such that $\rho(T^m x, T^m y) > c$.

For $n \in \mathbb{N}$, we define $\bar{x} = T^{m-n}x$ and $\bar{y} = T^{m-n}y$. Then, $\rho_T^{[-n+1,n-1]}(\bar{x}, \bar{y}) \leq c$, with $\rho(T^n \bar{x}, T^n \bar{y}) > c$. This means that $\text{range}(T^n) > n - 1$, and as a consequence $\text{range}(T^n) = n$. □

Corollary 2 *Let (X, T) be an infinite expansive system. Then we have $\mathfrak{D}(X, T) \cap \langle T \rangle = \{\text{id}\}$.*

7 Turing Machines as Dynamical Systems

Let us briefly look at a particular type of automorphisms in the subshift case. Specifically, we will take a look at automorphisms coming from Turing machines to better understand how the asymptotic range is related to the rate at which information is transmitted. These examples also serve to establish the existence of non-trivial (that is, of infinite order) range distorted automorphisms on a broad class of SFTs.

We assume some knowledge about the basics of complexity theory and Turing Machines (TM), for a complete reference see [13]. In the context of dynamical systems, there are several ways of representing a Turing Machine as a dynamical system. In this section we will use a model where the head moves presented in [11].

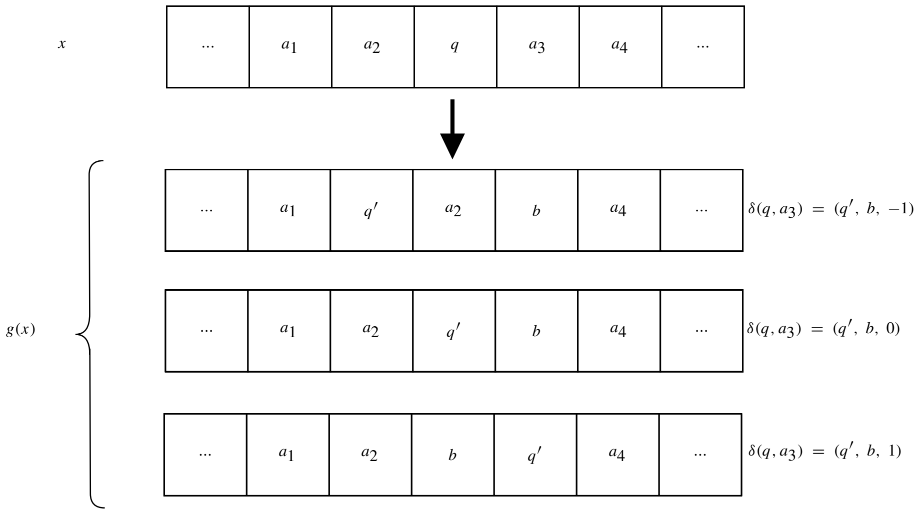
We will denote the set of states of the TM by Q, the alphabet by A and $\delta \colon Q \times A \to Q \times A \times \{-1, 0, 1\}$ its transition function.

For $n \geq 0$ we define the subshift,

$$X_n = \{x \in (Q \cup A)^{\mathbb{Z}} : |\{i \in \mathbb{Z} : x_i \in Q\}| \leq n\}.$$

It is possible to show that X_n is a sofic subshift. That is, can be obtained as a factor of a subshift of finite type. We do not need to be more precise in this article.

Definition 12 We define a moving head Turing Machine (TMH) as $g \in \mathrm{End}(X_1)$, where the head (given by the coordinate $x_i \in Q$) points to the site in its right and g executes the machine given by the transition function $\delta \colon Q \times A \to Q \times A \times \{-1, 0, 1\}$. It is easy to see that for all TMH g, range$(g) = 2$. See next figure for an illustration of this definition.



Definition 13 The position function $\mathfrak{p} \colon X_1 \to \mathbb{Z} \cup \{\infty\}$ of a TMH g is defined by $\mathfrak{p}(x) = n$ if $x_n \in Q$ and $\mathfrak{p}(x) = \infty$ on the other case.

The furtherest site the machine visits up to time t by the machine on configuration $x \in X_1$ is:

$$s_t(x) := \max\{|\mathfrak{p}(g^s(x))| : 0 \leq s \leq t\}.$$

Then, we define the movement function of the machine at time t as:

$$m(t) = \max_{x \in X_1} s_t(x).$$

It is clear that range$(g^t) = m(t)$. To carefully examine the possible growth rates of $m(t)$, we introduce the following asymptotic growth notation.

Definition 14 Let $\kappa, \eta \colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$. We write $\kappa(n) = O(\eta(n))$ if there exists a constant K such that $\kappa(n) \leq K\eta(n)$ for all sufficiently large n. Similarly, we write $\kappa(n) = \Omega(\eta(n))$ if $\eta(n) = O(\kappa(n))$. Furthermore, we write $\kappa(n) = \Theta(\eta(n))$ if $\kappa(n) = O(\eta(n))$ and $\kappa(n) = \Omega(\eta(n))$.

There exists a trichotomy with respect to the velocity that machines can have,

Theorem 6 ([7], Theorem 1) *Let g be a TMH with movement function m. Then, exactly one of the following holds:*

- *m is bounded,*
- *$m(t) = \Omega(\log(t))$ and $m(t) = O(t/\log(t))$,*
- *$m(t) = \Theta(t)$.*

In addition, it is possible to establish a connection between the periodicity of the function and its asymptotic speed rate.

Theorem 7 ([7], Theorem 2) *Every TMH with no weakly periodic configurations on $X_1 \setminus X_0$ is range distorted.*

An example of an aperiodic machine is constructed in [5]. Called the SMART machine, this reversible TMH is among other properties, aperiodic, which given the previous theorem implies that it is distorted. To find distorted automorphisms on the full-shift, we can embed this and other TMH's into its automorphism group through the use of conveyor belts. By slightly modifying Lemma 3 from [7] we obtain the following result:

Proposition 6 *Let g be a TMH. Then, by defining*

$$\Gamma = (\Sigma^2 \times \{<, >\}) \cup (Q \times \Sigma) \cup (\Sigma \times Q),$$

there exists an endomorphism $f \colon \Gamma^{\mathbb{Z}} \to \Gamma^{\mathbb{Z}}$ such that if $m \colon \mathbb{N} \to \mathbb{N}$ is the movement function of g, then $\mathrm{range}(f^t) \leq m(t)$ for all $t \in \mathbb{N}$. Furthermore, f is reversible if and only if g is.

Because the conveyor belt method allows us to see every reversible TMH within the automorphism group of a full-shift, we can conclude the following:

Theorem 8 *Let (X, σ) be a mixing SFT. Then, the set of reversible TMH is contained in $\mathrm{Aut}(X, \sigma)$. In particular, it contains an infinite order distorted automorphism.*

The proof of this fact follows directly from the previous proposition and the Kim-Roush Theorem for automorphisms groups of mixing shifts of finite type [10].

8 Non-shift Examples

Let us look at an example of the presented results for an expansive system that is not a subshift. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. We endow this space with a metric induced by the 2-norm on $\mathbb{R}^n$:

$$\rho(x, y) = \inf_{k\in\mathbb{Z}^n} \|x - y - k\|_2.$$

To find expansive homeomorphisms on this space, we use the following result about automorphisms of the torus.

Proposition 7 *Let T_A be an automorphism of the n-torus, with $n \geq 2$ and A its corresponding matrix on $GL(n, \mathbb{Z})$ over $\mathbb{R}^2$. Then, the following statements are equivalent:*

1. *T_A is expansive,*
2. *$A \in GL(n, \mathbb{Z})$ is expansive in $\mathbb{R}^n$,*
3. *A has no eigenvalue of modulus 1.*

A matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ is said to be expansive if there exists a constant $c > 1$ such that $\|Ax\| \geq c\|x\|$ for all $x \in \mathbb{R}^n$.

We note that n must be grater or equal than two, due to the fact that there are no expansive automorphisms on the 1-torus. The proof of this fact and of the proposition is outlined in [14]. By following its procedure, we can obtain the following lemma.

Lemma 13 *Let T_A be an expansive automorphism of the n-torus, where $n \geq 2$. Then, if we define $L'(A) = \max\{\|A\|, \|A^{-1}\|\}$, the expansive constant for the automorphism is given by*

$$c = \min\left\{\frac{1}{2L'(A)}, \frac{1}{4}\right\}.$$

Proof Due to the previous Proposition, we know that if T_A is expansive, A is expansive. This in turn means that the set $\{\|A^m x\| : m \in \mathbb{Z}\}$ is unbounded.

Because T_A is linear, we only have to prove the following: for $x \in \mathbb{T}^n$ such that $x \neq 0$, then there exists $m \in \mathbb{Z}$ such that $\rho(T_A^m x, 0) > c$. We do this in two cases. If $\|x\| > c$, it is evident that:

$$\rho(T_A^0 x, 0) = \|x\|_2 > c.$$

If $\|x\| \leq c$, due to the aforementioned set being unbounded we can define:

$$k = \inf\{|m| : \|A^m x\| > c,\ m \in \mathbb{Z}\}.$$

Let us suppose without loss of generality that $\|A^k x\| > c$. Then we have that

$$c < \|A^k x\| \leq \|A\| \|A^{k-1} x\| \leq L'(A) c \leq \frac{1}{2},$$

which means that $A^k x \in (-1, 1)^n$. Finally, $\rho(T_A^k x, 0) = \|A^k x\| > c$. □

Let us see how everything works by taking the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

For simplicity's sake we will use the same notation for A and T_A.

Its eigenvalues are $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$, which due to Proposition 7 means that it defines an expansive homeomorphism. It is possible to see that its expansive constant is in fact $c = \frac{1}{4}$.

Furthermore, by examining at the matrices that commute with A, we find that

$$\mathrm{Aut}(\mathbb{T}^2, A) = \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} : a^2 \neq 2b^2,\ a, b \in \mathbb{Z} \right\}.$$

It is possible to observe that a matrix such as

$$M = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \in \mathrm{Aut}(\mathbb{T}^2, A)$$

satisfies range$(M) = 1$, due to the fact that for $x, y \in \mathbb{T}^2$

$$\rho(Ax, Ay) \leq \frac{1}{4} \implies \rho(Mx, My) \leq \frac{1}{4}.$$

Finally, we notice that the eigenvalues of matrices in $\mathrm{Aut}(\mathbb{T}^2, A)$ are given by $\lambda_1 = a + \sqrt{2}b$ and $\lambda_2 = a - \sqrt{2}b$. By Theorem 4, we have that

$$AD(\mathbb{T}^2, A) = \{I, -I\},$$

where I is the identity matrix.

9 Conclusion

Throughout this work we have seen and developed the connections between distortion and non-expansivity. First, by generalizing the concept of range to general expansive systems, we have seen the aforementioned connection is not exclusive to subshifts. This ultimately led to the fact that expansive automorphisms cannot be distorted. What the obtained results suggest, is that a non-expansive direction is one in which the rate at which information propagates with sub-linear speed. Also, this generalization allows us to use concepts and tools from the study of automorphisms on symbolic systems to general expansive ones.

Nevertheless, the greatest question concerning distortion on automorphism groups remains open: is every range distorted automorphism group distorted? Even though there seems to be a direct path for solving this question, constructing a Turing machine-like automorphism that is range distorted but not group distorted, it is not clear how the construction of this automorphism can be achieved. It is possible that the study of the group generated by symmetrically distorted automorphisms, $\langle \mathrm{d}(X) \rangle$, can shed some light on this mystery. It can also be possible to answer the question by studying the group distorted Turing machines on the group of reversible Turing machines presented by Barbieri, Kari and Salo in [1]. There also remains to see if it is possible to have an automorphism with a unique non-expansive direction of irrational slope over a domain which is transitive or minimal. A possible approach consists on codifying a subshift suspension in a way that the non-expansive directions of the suspension are preserved.

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